FEM Homework Report

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1 Introduction

Make a program to solve the boundary value problem with purely Dirichlet boundary condition:

$$\begin{cases} -\Delta u = f, \quad x \in \Omega \\ u\big|_{\Gamma} = 0 \end{cases} \tag{1}$$

Use a triangular mesh and a piecewise-quadratic polynomial space V_h as the finite element space. Use $f(x, y) = 2(1-x)\sin x \cos y + 2(1-y)\sin y \cos x - 2(1-x)(1-y)\sin x \sin y$, $u(x, y) = (x-1)(y-1)\sin x \sin y$, $\Omega = (0, 1)^2$, $\Gamma = \partial\Omega$ to test the program, and compute the following errors:

$$\|u - u_h\|_{L^2(\Omega)}, \quad \|u - u_h\|_{H^1(\Omega)} \tag{2}$$

2 Method

2.1 The Galerkin approximation

We use a triangle mesh $\mathcal{T}_h$ to discretize the computational domain $\Omega_h = \Omega$, $\Gamma_h = \partial\Omega_h$ (since Ω is a polygon). Denote h to be the maximal arc-length of the elements as the characteristic length of the mesh. Define the piecewise-quadratic finite element space:

$$V_h = \left\{ v \in H^1(\Omega_h) \middle| v\big|_K \in \mathcal{P}^2(K), \forall K \in \mathcal{T}_h, v\big|_{\Gamma_h} = 0 \right\} \tag{3}$$

The Galerkin approximation to the problem using approximation space V_h is:

$$\begin{aligned} &\text{find } u_h \in V_h, \text{ such that:} \\ &\forall v_h \in V_h,\ \left(u_h', v_h'\right) = (f, v_h) \end{aligned} \tag{4}$$

2.2 The reference element

We use the right triangle as the reference element

$$\widehat{K} = \left\{ (x, y) \middle| 0 \leqslant x, y \leqslant x + y \leqslant 1 \right\} \tag{5}$$

with vertices $\widehat{P}_1 = (1, 0)$, $\widehat{P}_2 = (0, 1)$ and $\widehat{P}_3 = (0, 0)$. We add additional nodes on the edge: $\widehat{P}_4 = (0, \frac{1}{2})$, $\widehat{P}_5 = \left(\frac{1}{2}, 0\right)$, $\widehat{P}_6 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Within this reference element, the barycentric coordinates are given by $(\lambda_1, \lambda_2, \lambda_3) = (x, y, 1 - x - y)$ and the quadratic nodal bases for the above 6 nodes are given by:

$$\widehat{N}_1(x, y) = \lambda_1 (2\lambda_1 - 1) \tag{6a}$$

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$$\widehat{N}_2(x,y) = \lambda_2\left(2\lambda_2 - 1\right) \tag{6b}$$
$$\widehat{N}_3(x,y) = \lambda_3\left(2\lambda_3 - 1\right) \tag{6c}$$
$$\widehat{N}_4(x,y) = 4\lambda_2\lambda_3 \tag{6d}$$
$$\widehat{N}_5(x,y) = 4\lambda_3\lambda_1 \tag{6e}$$
$$\widehat{N}_6(x,y) = 4\lambda_1\lambda_2 \tag{6f}$$

Let $\widehat{\boldsymbol{N}}(x,y) = \left(\widehat{N}_1, \widehat{N}_2, \widehat{N}_3, \widehat{N}_4, \widehat{N}_5, \widehat{N}_6\right)$. The reference mass matrix is [1]

$$\widehat{\boldsymbol{M}} = \int_{\widehat{K}} \widehat{\boldsymbol{N}}(x,y)^T \widehat{\boldsymbol{N}}(x,y)\mathrm{d}x\mathrm{d}y = \left(\begin{array}{ccc|ccc} \frac{1}{60} & -\frac{1}{360} & -\frac{1}{360} & -\frac{1}{90} & 0 & 0 \\ -\frac{1}{360} & \frac{1}{60} & -\frac{1}{360} & 0 & -\frac{1}{90} & 0 \\ -\frac{1}{360} & -\frac{1}{360} & \frac{1}{60} & 0 & 0 & -\frac{1}{90} \\ \hline -\frac{1}{90} & 0 & 0 & \frac{4}{45} & \frac{2}{45} & \frac{2}{45} \\ 0 & -\frac{1}{90} & 0 & \frac{2}{45} & \frac{4}{45} & \frac{2}{45} \\ 0 & 0 & -\frac{1}{90} & \frac{2}{45} & \frac{2}{45} & \frac{4}{45} \end{array}\right) \tag{7}$$

The differentiation relations are [2]

$$\frac{\partial}{\partial x}\boldsymbol{N}(x,y) = \boldsymbol{N}(x,y)\begin{pmatrix} 3 & 0 & 1 & 0 & -4 & 0 \\ -1 & 0 & 1 & -4 & 0 & 4 \\ -1 & 0 & -3 & 0 & 4 & 0 \\ -1 & 0 & -1 & -2 & 2 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & -2 & 2 \end{pmatrix} = \boldsymbol{N}(x,y)\widehat{\boldsymbol{D}}_1 \tag{8a}$$

$$\frac{\partial}{\partial y}\boldsymbol{N}(x,y) = \boldsymbol{N}(x,y)\begin{pmatrix} 0 & -1 & 1 & 0 & -4 & 4 \\ 0 & 3 & 1 & -4 & 0 & 0 \\ 0 & -1 & -3 & 4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -2 & 2 \\ 0 & 1 & 1 & -2 & -2 & 2 \end{pmatrix} = \boldsymbol{N}(x,y)\widehat{\boldsymbol{D}}_2 \tag{8b}$$

2.3 The local bases and matrices

For an element $K = \Delta P_1P_2P_3$, where $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$, the Jacobianis

$$\boldsymbol{J}_K = \frac{\partial\left(x,y\right)}{\partial(\lambda_1,\lambda_2)} = \begin{pmatrix} x_i - x_k & x_j - x_k \\ y_i - y_k & y_j - y_k \end{pmatrix} \tag{9}$$

The **signed** area of the triangle is given by:

$$\Delta_K = \frac{1}{2}\det\left(\boldsymbol{J}_K\right) \tag{10}$$

Let $\boldsymbol{N}(x,y) = (N_1, N_2, N_3, N_4, N_5, N_6)$. The local mass matrix is:

$$\boldsymbol{M}_K = \left|\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right|\widehat{\boldsymbol{M}} = \det\left(\boldsymbol{J}_K\right)\widehat{\boldsymbol{M}} \tag{11}$$

The local differentiation relations are

$$\frac{\partial}{\partial x}\boldsymbol{N}(x,y) = \boldsymbol{N}(x,y)\left(\frac{\partial\lambda_1}{\partial x}\widehat{\boldsymbol{D}}_1 + \frac{\partial\lambda_2}{\partial x}\widehat{\boldsymbol{D}}_2\right) \tag{12a}$$
$$\frac{\partial}{\partial y}\boldsymbol{N}(x,y) = \boldsymbol{N}(x,y)\left(\frac{\partial\lambda_1}{\partial y}\widehat{\boldsymbol{D}}_1 + \frac{\partial\lambda_2}{\partial y}\widehat{\boldsymbol{D}}_2\right) \tag{12b}$$

[1] In fact, we use the `symbolic toolbox` in `matlab` to automatically generate this matrix.

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The local stiffness matrix is

$$
\begin{aligned}
\boldsymbol{S}_K &= \int_K (\nabla_{\boldsymbol{x}} \boldsymbol{N})^T \nabla_{\boldsymbol{x}} \boldsymbol{N} \mathrm{d}\boldsymbol{x} \\
&= \int_{\widehat{K}} \left(\frac{\partial(\lambda_1, \lambda_2)}{\partial(x,y)}^T \nabla_{\boldsymbol{\lambda}} \widehat{\boldsymbol{N}} \right)^T \left(\frac{\partial(\lambda_1, \lambda_2)}{\partial(x,y)}^T \nabla_{\boldsymbol{\lambda}} \widehat{\boldsymbol{N}} \right) \left| \frac{\partial(x,y)}{\partial(\lambda_1, \lambda_2)} \right| \mathrm{d}\boldsymbol{\lambda} \\
&= \left(\frac{\partial \lambda_1}{\partial x} \widehat{\boldsymbol{D}}_1 + \frac{\partial \lambda_2}{\partial x} \widehat{\boldsymbol{D}}_2 \right)^T \boldsymbol{M}_K \left(\frac{\partial \lambda_1}{\partial x} \widehat{\boldsymbol{D}}_1 + \frac{\partial \lambda_2}{\partial x} \widehat{\boldsymbol{D}}_2 \right) \\
&\quad + \left(\frac{\partial \lambda_1}{\partial y} \widehat{\boldsymbol{D}}_1 + \frac{\partial \lambda_2}{\partial y} \widehat{\boldsymbol{D}}_2 \right)^T \boldsymbol{M}_K \left(\frac{\partial \lambda_1}{\partial y} \widehat{\boldsymbol{D}}_1 + \frac{\partial \lambda_2}{\partial y} \widehat{\boldsymbol{D}}_2 \right) \\
&= \begin{pmatrix} \widehat{\boldsymbol{D}}_1^T & \widehat{\boldsymbol{D}}_2^T \end{pmatrix} \left(\left(\boldsymbol{J}_K^T \boldsymbol{J}_K \right)^{-1} \otimes \boldsymbol{M}_K \right) \begin{pmatrix} \widehat{\boldsymbol{D}}_1 \\ \widehat{\boldsymbol{D}}_2 \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\boldsymbol{D}}_1^T & \widehat{\boldsymbol{D}}_2^T \end{pmatrix} \left(\left(\boldsymbol{J}_K^T \boldsymbol{J}_K \right)^{-1} \otimes \det\left(\boldsymbol{J}_K \right) \widehat{\boldsymbol{M}} \right) \begin{pmatrix} \widehat{\boldsymbol{D}}_1 \\ \widehat{\boldsymbol{D}}_2 \end{pmatrix}
\end{aligned} \tag{13}
$$

To approximate integrals of non-polynomial functions, we apply the 3-point quadrature rule:

$$
\int_{\widehat{K}} f(x,y) \mathrm{d}\sigma \approx \left| \widehat{K} \right| \left(\frac{1}{3} f(\frac{1}{6}, \frac{1}{6}) + \frac{1}{3} f(\frac{1}{6}, \frac{2}{3}) + \frac{1}{3} f(\frac{2}{3}, \frac{1}{6}) \right) \tag{14}
$$

which has 2-order algebraic exactness.

Remark 1. *In order to make it easier to assembly the LHS matrix and RHS vector, we do not exclude boundary nodes from the total DOFs. For each j such that P_j is a boundary node, we enforce the j-th entry of the RHS vector, the j-th row and column of the LHS matrix (except the (j,j)-th entry) to be zero, and enforce the (j,j)-th entry of the LHS matrix to be 1. In this manner, the LHS matrix is kept symmetric positive definite.*

3 Results

We use the open-source mesh generation tool software `gmsh` to generate the meshes in the way of repeatedly refining the coarsest mesh. By running the scripts in `main.m`, the outputs from the `matlab` program we coded are listed below.

Our initial mesh and corresponding LHS matrix sparsity pattern are shown in the following figures:

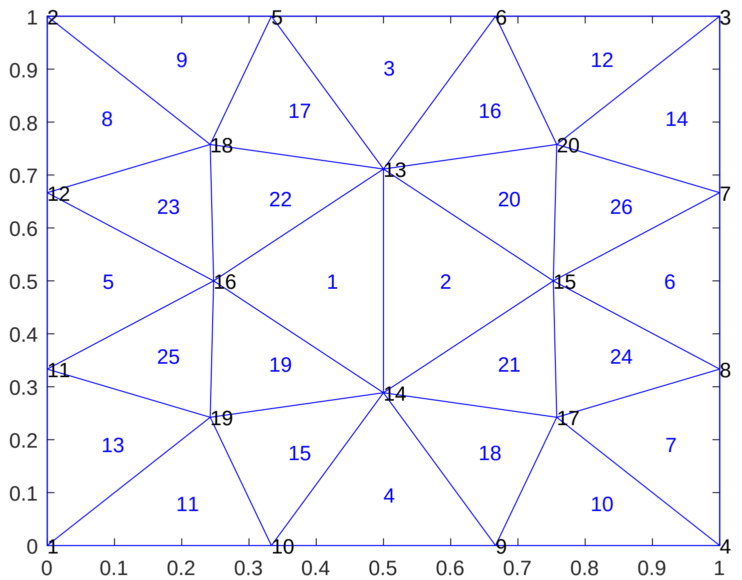

Figure 1: the coarsest mesh

The errors, meshes and sparsity pattern are shown below:

triangles	interior nodes	boundary nodes	L^2 error	order	H^1 error	order
26	41	24	3.3977e-04	-	5.3993e-03	-
104	185	48	4.3123e-05	2.9781	1.3155e-03	2.0372
416	785	96	5.4310e-06	2.9892	3.2698e-04	2.0083
1664	3233	192	6.8078e-07	2.9960	8.1659e-05	2.0015

Table 1: Table of error and order

Here, the nodes include all the collocation points used to compute nodal values.

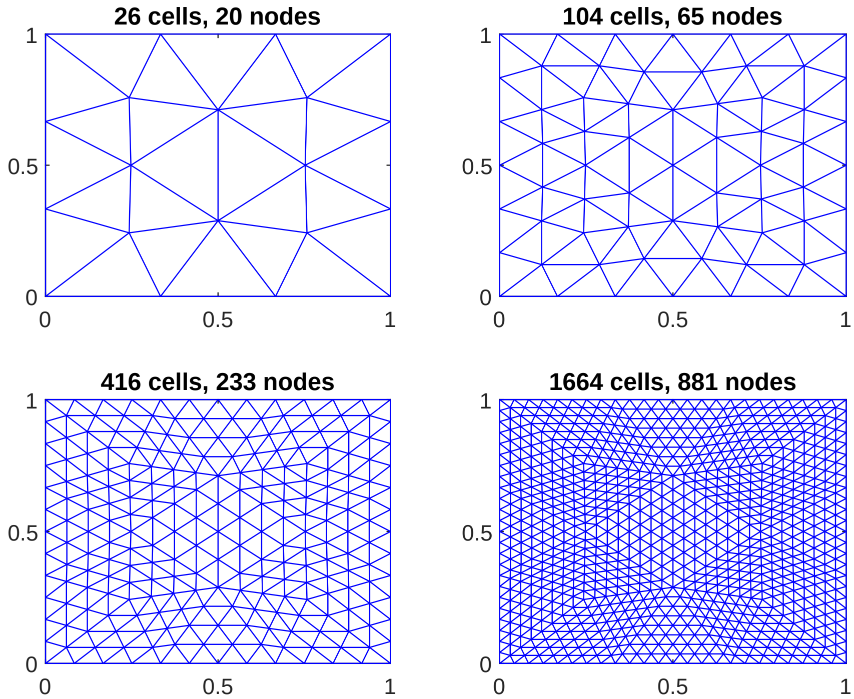

Figure 2: meshes

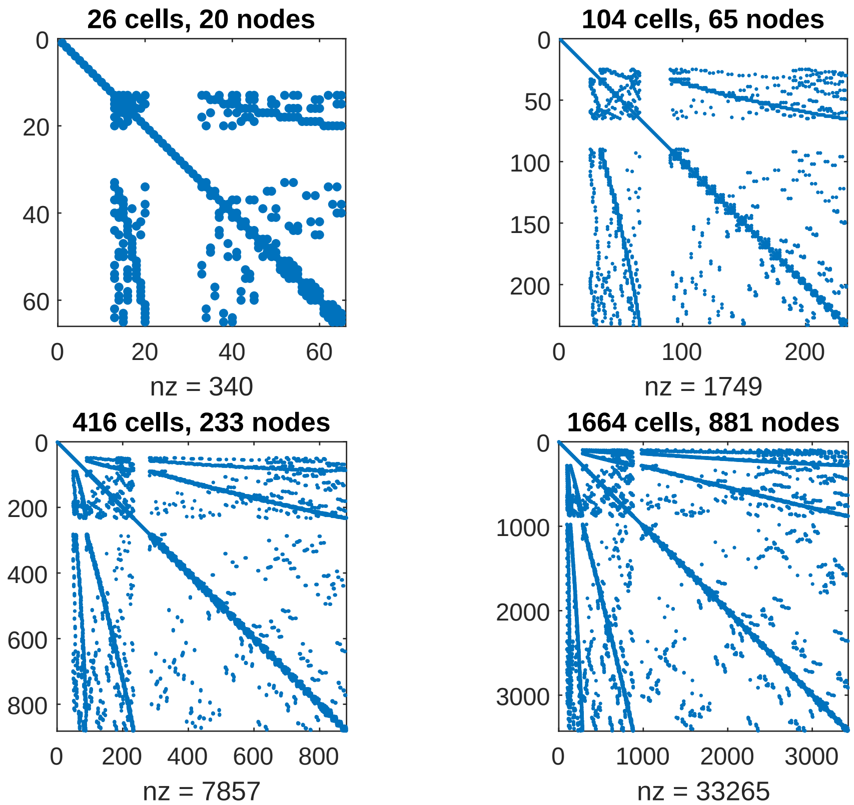

Figure 3: sparsity patterns

4 Discussion

From this experiement, we verified the theoretical 3-nd order convergence under the L^2 norm and 2-st order convergence under the H^1 norm of piecewise-quadratic H^1-conforming FEM numerically. The numerical convergence orders shown in the table seem to approach 3 and 2 from below, respectively.

We learned from the experiement that in higher dimensions, there are not collocation methods where the collocation points can serve as high-order quadrature points for exact integration of inner-products between basis polynomials. In this program, we used the special 3-point quadrature rule to approximate integration of non-polynomial functions in the right-hand-side. This does not cause the convergence order to degenerate since the quadrature has algebraic exactness of order 2. Meanwhile, the unstructure nature of the triangular mesh makes it harder for us to assemble the left-hand-side matrix, especially in the case of Dirichlet BCs, thus we need to adapt new strategies to enforce such boundary condition. Last but not least, when testing FEM programs using unstructured grids, we have to refine the coarse mesh by a factor of 2 in order to preserve similarity between elements in most of the cases. As a result, the computational cost could be prohibitive after several refinement and our test numbers might be limited.