WENO 算法、实现与讨论

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目录

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第一部分 Weighted Essentially Non-oscillatory Scheme

1 Space descretization

定义：

- 网格：$a = x_{\frac{1}{2}} < a_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$.
- 单元：$I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$，单元中点：$x_i = \frac{x_{i-\frac{1}{2}}+x_{i+\frac{1}{2}}}{2}$，单元大小：$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$.
- 网格大小：$\Delta x = \min\limits_{1\leqslant i\leqslant N} \Delta x_i$.

我们先考虑标量情形，且不考虑时间离散和边界问题。

1.1 由点值插值点值

给定插值误差阶 k 以及点值 $u_j = u\left(x_j\right), 1 \leqslant j \leqslant N$，使用多项式对 $u(x), x \in I_i$ 进行插值。

为了减少判断次数以及充分利用 $2k-1$ 个候选点构成的全部 k 个候选模板，我们使用上面 k 个模板提供的近似 $u_x^{(r)}, 0 \leqslant r \leqslant k-1$ 的**凸组合**来近似，目标是达到尽可能高的精度。

当采用等距节点时，若 $u \in C^{2k-1}\left[x_{i-r}, x_{i-r+k-1}\right]$，一定存在凸组合系数 $d_r \geqslant 0$，$\sum\limits_{r=0}^{k-1} d_r = 1$ 使得：

$$u(x) = \sum_{r=0}^{k-1} d_r u^r(x) + \mathcal{O}\left(\Delta x^{2k-1}\right) \tag{1}$$

而当非等距节点时，如果使用 Taylor 展开以及待定系数法，可能存在负系数 $d_r < 0$，这时不能保持格式的的单调性，会发生震荡，需要特殊处理. 但是，如果任意取凸组合系数 d_r，总可以使得逼近具有 $\mathcal{O}\left(\Delta x^k\right)$ 的误差阶，因此我们总可以试着达到更高的误差阶。

假设我们已经有凸组合系数 $d_r \geqslant 0$ 使得具有 $2k-1$ 阶收敛阶。由 Godunov 定理，单调 (线性) 格式至多具有 1 阶局部截断误差，因此我们需要选取非线性系数 ω_r 使得 $\omega_r \geqslant 0$，$\sum\limits_{r=0}^{k-1} \omega_r = 1$，且当 u 足够光滑时，$\omega_r = d_r + \mathcal{O}\left(\Delta x^{k-1}\right)$. 这是由于考虑到需要 u 足够光滑 ($2k-1$ 阶连续可微) 时局部截断误差为 $\mathcal{O}\left(\Delta x^{2k-1}\right)$.

我们取 $\omega_r = \frac{\alpha_r}{\sum\limits_{j=0}^{k-1}\alpha_j}$，其中 $\alpha_r = \frac{d_r}{(\epsilon+\beta_r)^2}$，其中 $\epsilon = 10^{-6}$(为了防止分母为 0)，β_r 为光滑因子：当 u 光滑时，$\omega_r \approx d_r$；当 u 不光滑时，$\omega_r \approx 0$. 注意到 $\deg p_{i,r} \leqslant k-1$，我们可以选取：

$$\beta_r = \sum_{l=1}^{k-1} \Delta x_i^{2l-1} \int_{I_i} \left(\frac{\mathrm{d}^l}{\mathrm{d}x^l} p_{i,r}(x)\right)^2 \mathrm{d}x \tag{2}$$

最后构造：

$$p_i(x) = \sum_{r=0}^{k-1} \omega_r p_{i,r}(x) \tag{3}$$

为最终的插值多项式。

注记：当 u 全局光滑时，该格式是收敛的，且若使用等距节点，格式 TVB 且拥有 $2k-1$ 阶局部截断误差。

1.2 由单元均值重构点值

定义单元均值：

$$\overline{u}_i = \frac{1}{\Delta x_i} \int_{I_i} u(x) \mathrm{d}x \tag{4}$$

并且此时的模板按照区间而非结点来写。

我们的目标：利用单元均值来重构点值。

设 $U(x)$ 为 $u(x)$ 的原函数，也即：

$$U(x) = \int_{-\infty}^{x} u(t) \mathrm{d}t \tag{5}$$

设 $P_{i,r}(x)$ 是对 $U(x)$ 在对应于 k 个区间的模板

$$S_i^{(r)} = \left\{I_{i-r}, \cdots, I_{i-r+k-1}\right\}$$

的 $k+1$ 点模板

$$\widehat{S}_i^{(r)} = \left\{x_{i-r-\frac{1}{2}}, \cdots, x_{i-r+k-\frac{1}{2}}\right\}$$

上的至多 k 次插值多项式，那么：

- $p_{i,r}(x)=\frac{\mathrm{d}}{\mathrm{d}x}P_{i,r}(x)$ 和 $u(x)$ 在 $I_j,\ i-1\leqslant j\leqslant i-r+k-1$ 上的单元均值相等。
- $U(x)=P_{i,r}(x)+\mathcal{O}\left(\Delta x^{k+1}\right)$，若 $u\in C^k$.
- $u(x)=p_{i,r}(x)+\mathcal{O}\left(\Delta x^{k}\right)$，若 $u\in C^k$ 且 x 是区间端点 $x_{i\pm\frac{1}{2}}$ (由多项式插值误差定理).

同时，注意到：

$$U\left[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}\right]=\overline{u}_j \tag{6}$$

且 U 的常数项可以不妨设为 0，所以我们可以利用与上文类似的方式，用 ENO/WENO 自适应地计算插值多项式，利用符号求导，即可得到边界点值 $u^+_{i-\frac{1}{2}}$ 与 $u^-_{i+\frac{1}{2}}$.

1.3 由点值插值数值通量

目的：给定 u 在单元中点 x_j 上的取值 u_j，求数值通量 $\widehat{u}_{i+\frac{1}{2}}=\widehat{u}\left(u_{i-r},\cdots,u_{i-r+k-1}\right)$，使得：

$$\frac{1}{\Delta x_i}\left(\widehat{u}_{i+\frac{1}{2}}-\widehat{u}_{i-\frac{1}{2}}\right)=u'\left(x_i\right)+\mathcal{O}\left(\Delta x^k\right) \tag{7}$$

注记：由下文可见，在用点值逼近数值通量时，我们必须使用均匀网格 (或光滑网格)。

注意到：如果存在函数 $h(x)$ 使得

$$u(x)=\frac{1}{\Delta x}\int_{I_i}h(\xi)\mathrm{d}\xi=\overline{h}_i \tag{8}$$

那么：

$$u'(x)=\frac{1}{\Delta x}\left(h\left(x+\frac{\Delta x}{2}\right)-h\left(x-\frac{\Delta x}{2}\right)\right) \tag{9}$$

于是只需要

$$u'(x_i)=\frac{1}{\Delta x}\left(h_{i+\frac{1}{2}}-h_{i-\frac{1}{2}}\right)+\mathcal{O}\left(\Delta x^k\right) \tag{10}$$

$$\widehat{u}_{i+\frac{1}{2}}=h_{i+\frac{1}{2}}+\mathcal{O}\left(\Delta x^k\right) \tag{11}$$

即可，这是因为误差部分经常是光滑的。

此时代入 $\overline{h}_j=u_j$，用上文的 ENO/WENO 格式在 I_i 上选取模板，重构 $h^+_{i-\frac{1}{2}}$ 和 $h^-_{i+\frac{1}{2}}$ 即可。

1.4 WENO 插值系数

1.4.1 固定模板的插值系数

给定误差阶 $k\in\mathbb{N}_+$，设每一个模板 $S_i^{(r)}=\{x_{i-r},\cdots,x_{i-r+k-1}\}$ 在 $x\in I_i$ 处的插值结果为：

$$u_{i,r}(x)=\sum_{j=0}^{k-1}c_{i,r,j}(x)u_{i-r+j}+\mathcal{O}\left(\Delta x^k\right) \tag{12}$$

其中 $0\leqslant r\leqslant k-1$，$c_{i,r,j}(x)\in\mathbb{R}_k[x]$. 由 Lagrange 多项式插值公式得：

$$c_{i,r,j}(x)=\prod_{\substack{m=0\\m\neq j}}^{k-1}\frac{x-x_{i-r+m}}{x_{i-r+j}-x_{i-r+m}} \tag{13}$$

1.4.2 WENO 插值线性权

对于 $u(x)$ 在上文中的 k 个模板的重构结果 $u_{i,r}(x),\ 0\leqslant r\leqslant k-1$，由插值多项式的唯一性知，存在唯一的系数 (仅与网格有关) $d_{i,r}(x)$ 使得：

$$u_i(x)=\sum_{r=0}^{k-1}d_{i,r}(x)u_{i,r}(x)=\sum_{r=0}^{k-1}\sum_{j=0}^{k-1}d_{i,r}(x)c_{i,r,j}(x)u_{i-r+j} \tag{14}$$

在 $\bigcup_{r=0}^{k-1}S_i^{(r)}$ 上插值 $u(x)$.

注意到：上式相当于使用大模板 $\widehat{S}_i=\{x_{i-k+1},\cdots,x_{i+k-1}\}$ 进行 $2k-1$ 阶插值的结果。因此，设该大模板插值系数为 $\widehat{c}_{i,j}(x)\in\mathbb{R}_{2k-1}[x],\ 0\leqslant j\leqslant 2k-2$ (相当于使用 $2k-1$ 阶，左移量为 $k-1$ 模板的系数)，那么对于 $0\leqslant j\leqslant 2k-2$，有：

$$\sum_{\max\{0,k-j-1\}\leqslant r\leqslant\min\{2k-2-j,k-1\}}c_{i,r,r+j-k+1}(x)d_{i,r}(x)=\widehat{c}_{i,j}(x) \tag{15}$$

也即，对于 $0\leqslant j\leqslant 2k-2$：

$$\begin{aligned}&\sum_{\max\{0,k-j-1\}\leqslant r\leqslant\min\{2k-2-j,k-1\}}\prod_{\substack{m=k-r-1\\ m\neq j}}^{2k-2-r}\frac{x-x_{i-k+m+1}}{x_{i-k+j+1}-x_{i-k+m+1}}d_{i,r}(x)\\ =&\prod_{\substack{m=0\\ m\neq j}}^{2k-2}\frac{x-x_{i-k+m+1}}{x_{i-k+j+1}-x_{i-k+m+1}}\end{aligned} \tag{16}$$

由 Neville 算法知，因为 $u_{i,r}(x)$ 在模板 $S_i^{(r)}$ 上插值 $u(x)$，且 $\bigcup_{r=0}^{k-1}S_i^{(r)}=\widehat{S}_i$，所以一定存在多项式 $\widetilde{d}_{i,r}(x)$ 使得 $\sum\limits_{r=0}^{k-1}\widetilde{d}_{i,r}u_{i,r}(x)$ 在 $\widehat{S}_i$ 上插值 $u(x)$. 由唯一性，$d_{i,r}(x)\in\mathbb{R}_k[x]$. 具体地，我们下面给出 $d_{i,r}(x)$ 和 $u_i(x)$ 的构造过程。

设 $p_{i,m}^d(x)\in\mathbb{R}_{k+d}[x]$ 为在 $\bigcup_{r=m}^{m+d}S_i^{(r)}$ 上插值 $u(x)$ 的多项式，$0\leqslant m\leqslant m+d\leqslant k-1$，我们有以下递推式：

$$p_{i,m}^0(x)=u_{i,m}(x),\quad 0\leqslant m\leqslant k-1 \tag{17}$$

$$p_{i,m}^{d+1}(x)=\frac{x-x_{i-m-d-1}}{x_{i-m+k-1}-x_{i-m-d-1}}p_{i,m}^d(x)+\frac{x_{i-m+k-1}-x}{x_{i-m+k-1}-x_{i-m-d-1}}p_{i,m+1}^d(x) \tag{18}$$

易见，取 $u_i(x)=p_{i,0}^{k-1}(x)$ 即可。此时我们还可以给出 $d_{i,r}(x)$ 的一般表达式：

记因子：

$$\lambda_{i,m,t}(x)=\begin{cases}1, & t=1\\ \frac{x-x_{i+m}}{x_{i+m+t}-x_{i+m}}, & m<0,t\neq 1\\ \frac{x_{i+m}-x}{x_{i+m}-x_{i+m-t}}, & m>0,t\neq 1\end{cases} \tag{19}$$

那么：

$$d_{i,r}(x)=\begin{cases}\sum\limits_{\pi_r}\prod\limits_{j=1}^{k-1}\lambda_{i,m_j,k+j-1}(x), & k\geqslant 2\\ 1, & k=1\end{cases} \tag{20}$$

其中 $\pi_r = \{m_1, \cdots, m_{k-1}\}$ 取遍 $\{k-1, \cdots, k-1-r; -k+1, \cdots, -r\}$ 满足相同符号的数按绝对值递增的排列，共 $\binom{k-1}{r}$ 个。例如：$\{-2,4,-3,-4\}, \{-3,-4,3,4\}, \{2,3,-3\}$.

k	order	线性权 $d_{i,r}(x)$
1	1	$d_{i,0}(x) = 1$
2	3	$d_{i,0}(x) = \frac{x-x_{i-1}}{x_{i+1}-x_{i-1}}$ $d_{i,1}(x) = \frac{x_{i+1}-x}{x_{i+1}-x_{i-1}}$
3	5	$d_{i,0}(x) = \frac{x-x_{i-2}}{x_{i+2}-x_{i-2}} \frac{x-x_{i-1}}{x_{i+2}-x_{i-1}}$ $d_{i,1}(x) = \frac{x-x_{i-2}}{x_{i+2}-x_{i-2}} \frac{x_{i+2}-x}{x_{i+2}-x_{i-1}} + \frac{x_{i+2}-x}{x_{i+2}-x_{i-2}} \frac{x-x_{i-2}}{x_{i+1}-x_{i-2}}$ $d_{i,2}(x) = \frac{x_{i+2}-x}{x_{i+2}-x_{i-2}} \frac{x_{i+1}-x}{x_{i+1}-x_{i-2}}$
4	7	$d_{i,0}(x) = \frac{x-x_{i-3}}{x_{i+3}-x_{i-3}} \frac{x-x_{i-2}}{x_{i+3}-x_{i-2}} \frac{x-x_{i-1}}{x_{i+3}-x_{i-1}}$ $d_{i,1}(x) = \frac{x-x_{i-3}}{x_{i+3}-x_{i-3}} \frac{x-x_{i-2}}{x_{i+3}-x_{i-2}} \frac{x_{i+3}-x}{x_{i+3}-x_{i-1}} + \frac{x-x_{i-3}}{x_{i+3}-x_{i-3}} \frac{x_{i+3}-x}{x_{i+3}-x_{i-2}} \frac{x-x_{i-2}}{x_{i+2}-x_{i-2}} + \frac{x_{i+3}-x}{x_{i+3}-x_{i-3}} \frac{x-x_{i-3}}{x_{i+2}-x_{i-3}} \frac{x-x_{i-2}}{x_{i+2}-x_{i-2}}$ $d_{i,2}(x) = \frac{x-x_{i-3}}{x_{i+3}-x_{i-3}} \frac{x_{i+3}-x}{x_{i+3}-x_{i-2}} \frac{x_{i+2}-x}{x_{i+2}-x_{i-2}} + \frac{x_{i+3}-x}{x_{i+3}-x_{i-3}} \frac{x-x_{i-3}}{x_{i+2}-x_{i-3}} \frac{x_{i+2}-x}{x_{i+2}-x_{i-2}} + \frac{x_{i+3}-x}{x_{i+3}-x_{i-3}} \frac{x_{i+2}-x}{x_{i+2}-x_{i-3}} \frac{x-x_{i-3}}{x_{i+1}-x_{i-3}}$ $d_{i,3}(x) = \frac{x_{i+3}-x}{x_{i+3}-x_{i-3}} \frac{x_{i+2}-x}{x_{i+2}-x_{i-3}} \frac{x_{i+1}-x}{x_{i+1}-x_{i-3}}$

表 1: $k = 1, \cdots, 4$ 时的线性权 $d_{i,r}(x)$

同时，$d_{i,r}(x)$ 满足 $(2k-1) \times k$ 的超定方程组：

$$\boldsymbol{A}_i(x) \boldsymbol{d}_i(x) = \widehat{\boldsymbol{c}}_i(x) \tag{21}$$

其中 $\boldsymbol{d}_i(x) = (d_{i,0}, \cdots, d_{i,k-1})^T(x)$，$\widehat{\boldsymbol{c}}_i(x) = (\widehat{c}_{i,0}, \cdots, \widehat{c}_{i,2k-2})^T(x)$，

$$\boldsymbol{A}_i(x) = \begin{pmatrix} & & & & c_{i,k-1,0}(x) \\ & & & c_{i,k-2,0}(x) & c_{i,k-1,1}(x) \\ & & \iddots & \iddots & \vdots \\ & c_{i,1,0}(x) & \iddots & \iddots & c_{i,k-1,k-2}(x) \\ c_{i,0,0}(x) & c_{i,1,1}(x) & \iddots & c_{i,k-2,k-2}(x) & c_{i,k-1,k-1}(x) \\ c_{i,0,1}(x) & \iddots & \iddots & c_{i,k-2,k-1}(x) & \\ \vdots & \iddots & \iddots & & \\ c_{i,0,k-2}(x) & c_{i,1,k-1}(x) & & & \\ c_{i,0,k-1}(x) & & & & \end{pmatrix} \in \mathbb{R}_{k-1}[x]^{(2k-1)\times k} \tag{22}$$

注记：虽然方程组过定，但是根据 $c_{i,r,j}(x)$ 的定义，知该方程组一定存在唯一解，因此只需要求解一个 $k \times k$ 的反三角子方程组即可。

1.5 WENO 重构系数

1.5.1 固定模板的重构系数

给定误差阶 $k \in \mathbb{N}_+$，设每一个模板 $S_i^{(r)} = \{x_{i-r}, \cdots, x_{i-r+k-1}\}$ 在 $x \in I_i$ 处的重构结果为：

$$u_{i,r}(x) = \sum_{j=0}^{k-1} c_{i,r,j}(x) \overline{u}_{i-r+j} + \mathcal{O}\left(\Delta x^k\right) \tag{23}$$

其中 $0 \leqslant r \leqslant k-1$，$c_{i,r,j}(x) \in \mathbb{R}_{k-1}[x]$. 直接计算得：

$$c_{i,r,j}(x) = \sum_{m=j+1}^{k} \frac{\sum\limits_{\substack{l=0 \\ l\neq m}}^{k} \prod\limits_{\substack{q=0 \\ q\neq m,l}}^{k} \left(x - x_{i-r+q-\frac{1}{2}}\right)}{\prod\limits_{\substack{l=0 \\ l\neq m}}^{k} \left(x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}}\right)} \Delta x_{i-r+j} \tag{24}$$

1.5.2 WENO 重构线性权

对于 $u(x)$ 在上文中的 k 个模板的重构结果 $u_{i,r}(x)$, $0 \leqslant r \leqslant k-1$，由插值多项式的唯一性知，存在唯一的系数 (仅与网格有关，为有理函数) $d_{i,r}(x)$ 使得：

$$u(x) = \sum_{r=0}^{k-1} d_{i,r}(x) u_{i,r}(x) = \sum_{r=0}^{k-1}\sum_{j=0}^{k-1} d_{i,r}(x) c_{i,r,j}(x) \overline{u}_{i-r+j} \tag{25}$$

注意到：上式相当于使用大模板 $\widehat{S}_i = \{x_{i-k+1}, \cdots, x_{i+k-1}\}$ 进行 $2k-1$ 阶重构的结果。因此，设该大模板重构系数为 $\widehat{c}_{i,j}(x)$, $0 \leqslant j \leqslant 2k-2$ (相当于使用 $2k-1$ 阶，左移量为 $k-1$ 模板的系数)，那么：

$$\begin{aligned}\sum_{j=0}^{2k-2} \widehat{c}_{i,j}(x)\overline{u}_{i-k+j+1} &= \sum_{r=0}^{k-1}\sum_{j=0}^{k-1} d_{i,r}(x) c_{i,r,j}(x)\overline{u}_{i-r+j} \\ &= \sum_{j=0}^{2k-2} \sum_{\max\{0,k-j-1\}\leqslant r\leqslant \min\{2k-2-j,k-1\}} d_{i,r}(x) c_{i,r,r+j-k+1}(x)\overline{u}_{i-k+j+1}\end{aligned} \tag{26}$$

$$\Longrightarrow \sum_{\max\{0,k-j-1\}\leqslant r\leqslant \min\{2k-2-j,k-1\}} c_{i,r,r+j-k+1}(x) d_{i,r}(x) = \widehat{c}_{i,j}(x), \quad 0 \leqslant j \leqslant 2k-2 \tag{27}$$

因此 $d_{i,r}(x)$ 满足 $(2k-1)\times k$ 的超定方程组：

$$\boldsymbol{A}_i(x)\boldsymbol{d}_i(x) = \widehat{\boldsymbol{c}}_i(x) \tag{28}$$

其中 $\boldsymbol{d}_i(x) = (d_{i,0}, \cdots, d_{i,k-1})^T(x)$，$\widehat{\boldsymbol{c}}_i(x) = (\widehat{c}_{i,0}, \cdots, \widehat{c}_{i,2k-2})^T(x)$，

$$\boldsymbol{A}_i(x) = \begin{pmatrix} & & & & c_{i,k-1,0}(x) \\ & & & c_{i,k-2,0}(x) & c_{i,k-1,1}(x) \\ & & \iddots & \iddots & \vdots \\ & c_{i,1,0}(x) & \iddots & \iddots & c_{i,k-1,k-2}(x) \\ c_{i,0,0}(x) & c_{i,1,1}(x) & \iddots & c_{i,k-2,k-2}(x) & c_{i,k-1,k-1}(x) \\ c_{i,0,1}(x) & \iddots & \iddots & c_{i,k-2,k-1}(x) & \\ \vdots & \iddots & \iddots & & \\ c_{i,0,k-2}(x) & c_{i,1,k-1}(x) & & & \\ c_{i,0,k-1}(x) & & & & \end{pmatrix} \in \mathbb{R}_{k-1}[x]^{(2k-1)\times k} \tag{29}$$

注记：虽然方程组过定，但是根据 $c_{i,r,j}(x)$ 的定义，知该方程组一定存在唯一解，因此只需要求解一个 $k\times k$ 的反三角子方程组即可。

2 Temporal discretization

使用 TVD/TVB Runge-Kutta 进行时间离散，设半离散为算子形式：$u_t = L(u) = \mathcal{L}(u) + \mathcal{O}\left(\Delta x^k\right)$(误差当 u 光滑时成立).

目标：找到 k 阶的 Runge-Kutta TVD 时间离散。

设单步 Euler 向前格式在条件 $\Delta t \leqslant \Delta t_1$ 时是 TVD 的 ($\|\cdot\|_{TV}$ 算子范数不超过 1)。在 $\Delta t \leqslant c\Delta t_1$ 的条件下 (c 为 CFL 数)，考虑对时间的高阶 Runge-Kutta 格式。

一般地：

$$u^{(i)} = \sum_{k=0}^{i-1}\left(\alpha_{ik}u^{(k)} + \Delta t\beta_{ik}L\left(u^{(k)}\right)\right), \quad 1 \leqslant i \leqslant m \tag{30}$$

$$u^{(0)} = u^n, \quad u^{(m)} = u^{n+1} \tag{31}$$

如果系数 $\alpha_{ik}, \beta_{ik} \geqslant 0$，那么就是 Euler 单步向前的凸组合。由相容性知，$\sum\limits_{k=0}^{i-1}\alpha_{ik} = 1$.

因为计算 $u^{(i)}$ 时的 Δt 等价于被 $\frac{\beta_{ik}}{\alpha_{ik}}\Delta t$ 取代，所以由 CFL 条件知，当 $c \leqslant \min\limits_{i,k}\frac{\alpha_{i,k}}{|\beta_{i,k}|}$ 才能稳定，我们可以取 $c = \min\limits_{i,k}\frac{\alpha_{i,k}}{|\beta_{i,k}|}$.

若 $\beta_{i,k} < 0$，那么需要使用 L 的伴随算子 $\widetilde{L}$(向后 Euler 且 TVD)，其对应向后方程：$u_t = f(u)_x$. 易见，计算 L 和 $\widetilde{L}$ 时只有迎风方向正好相反，且两者给出的都是对 $f(u)_x$ 的近似。

- TVD Runge-Kutta 2，$c = 1$:

$$u^{(1)} = u^n + \Delta tL\left(u^n\right) \tag{32}$$

$$u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta tL\left(u^{(1)}\right) \tag{33}$$

- TVD Runge-Kutta 3，$c = 1$:

$$u^{(1)} = u^n + \Delta tL\left(u^n\right) \tag{34}$$

$$u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta tL\left(u^{(1)}\right) \tag{35}$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta tL\left(u^{(2)}\right) \tag{36}$$

- TVD Runge-Kutta 4，$c = 0.936$.

第二部分 Hyperbolic Conservation Law

考虑双曲守恒律问题：

$$\frac{\partial}{\partial t}u + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{F}(u) = 0 \tag{37}$$

3 One dimensional WENO procedure

将区间 $[a, b]$ 等距划分成 N 个区间：

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b \tag{38}$$

$$x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}, \quad I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right], \quad \Delta x_i = \Delta x, \quad i = 1, \cdots, N \tag{39}$$

3.1 3 阶 WENO 重构过程

问题：已知 $u(x)$ 在 x_i 上的取值 u_i，对数值通量 $\widehat{u}_{i+\frac{1}{2}}$ 重构，使得当 u 足够光滑时：

$$\frac{\widehat{u}_{i+\frac{1}{2}}-\widehat{u}_{i-\frac{1}{2}}}{\Delta x}=u'(x_i)+\mathcal{O}\left(\Delta x^3\right) \tag{40}$$

算法：利用 3 阶 WENO 分别从左右侧在 $x_{i+\frac{1}{2}}$ 近似数值通量 $\widehat{u}^{\pm}_{i+\frac{1}{2}}$.

1. 输入点值 $u_{i-1}, u_i, u_{i+1}, u_{i+2}$.
2. 在 $x_{i+\frac{1}{2}}$ 处计算 4 个固定 2 点模板近似：

$$\begin{aligned}
\widehat{u}^{-,(0)}_{i+\frac{1}{2}}&=\frac{1}{2}u_i+\frac{1}{2}u_{i+1}, \quad \widehat{u}^{-,(1)}_{i+\frac{1}{2}}=-\frac{1}{2}u_{i-1}+\frac{3}{2}u_i\\
\widehat{u}^{+,(0)}_{i+\frac{1}{2}}&=\frac{1}{2}u_{i+1}+\frac{1}{2}u_i, \quad \widehat{u}^{+,(1)}_{i+\frac{1}{2}}=-\frac{1}{2}u_{i+2}+\frac{3}{2}u_{i+1}
\end{aligned}$$

3. 计算光滑因子，合成非线性权：

$$\begin{aligned}
\beta_0=(u_i-u_{i+1})^2, \quad \beta_1=(u_{i-1}-u_i)^2; &\qquad \widetilde{\beta}_0=(u_{i+1}-u_i)^2, \quad \widetilde{\beta}_1=(u_{i+2}-u_{i+1})^2\\
\alpha_0=\frac{\frac{2}{3}}{(\epsilon+\beta_0)^2}, \quad \alpha_1=\frac{\frac{1}{3}}{(\epsilon+\beta_1)^2}; &\qquad \widetilde{\alpha}_0=\frac{\frac{2}{3}}{(\epsilon+\widetilde{\beta}_0)^2}, \quad \widehat{\alpha_1}=\frac{\frac{1}{3}}{(\epsilon+\widetilde{\beta}_1)^2}\\
\omega_0=\frac{\alpha_0}{\alpha_0+\alpha_1}, \quad \omega_1=\frac{\alpha_1}{\alpha_0+\alpha_1}; &\qquad \widehat{\omega}_0=\frac{\widetilde{\alpha}_0}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}, \quad \widetilde{\omega}_1=\frac{\widetilde{\alpha}_1}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}
\end{aligned}$$

4. 合成对数值通量的 3 阶 WENO 近似：

$$\widehat{u}^-_{i+\frac{1}{2}}=\omega_0\widehat{u}^{-,(0)}_{i+\frac{1}{2}}+\omega_1\widehat{u}^{-,(1)}_{i+\frac{1}{2}}; \qquad \widehat{u}^+_{i+\frac{1}{2}}=\widetilde{\omega}_0\widehat{u}^{+,(0)}_{i+\frac{1}{2}}+\widetilde{\omega}_1\widehat{u}^{+,(1)}_{i+\frac{1}{2}}$$

注记[1]：

- 3 阶 WENO 格式存在收敛阶不稳定 (递增) 的现象。
- 3 阶 WENO 格式的收敛阶与非线性权中 ϵ 的取值方式有一定关系 (如 $\epsilon=\Delta x^2$ 在 $f(u)\equiv u$ 时表现较好).

3.2 5 阶 WENO 重构过程

与 3 阶 WENO 类似，我们下面直接写出 5 阶 WENO 格式重构过程。其中，每一个 $\widehat{u}_i$ 的计算利用到了 3 个 3 点模板。

固定模板近似结果：

$$\widehat{u}^{-,(0)}_{i+\frac{1}{2}}=\frac{1}{3}u_i+\frac{5}{6}u_{i+1}-\frac{1}{6}u_{i+2}, \quad \widehat{u}^{+,(0)}_{i+\frac{1}{2}}=\frac{1}{3}u_{i+1}+\frac{5}{6}u_i-\frac{1}{6}u_{i-1} \tag{41}$$

$$\widehat{u}^{-,(1)}_{i+\frac{1}{2}}=-\frac{1}{6}u_{i-1}+\frac{5}{6}u_i+\frac{1}{3}u_{i+1}, \quad \widehat{u}^{+,(1)}_{i+\frac{1}{2}}=-\frac{1}{6}u_{i+2}+\frac{5}{6}u_{i+1}+\frac{1}{3}u_i \tag{42}$$

$$\widehat{u}^{-,(2)}_{i+\frac{1}{2}}=\frac{1}{3}u_{i-2}-\frac{7}{6}u_{i-1}+\frac{11}{6}u_i, \quad \widehat{u}^{+,(2)}_{i+\frac{1}{2}}=\frac{1}{3}u_{i+3}-\frac{7}{6}u_{i+2}+\frac{11}{6}u_{i+1} \tag{43}$$

线性权构造：

$$\beta_0=\frac{13}{12}\left(u_i-2u_{i+1}+u_{i+2}\right)^2+\frac{1}{4}\left(3u_i-4u_{i+1}+u_{i+2}\right)^2 \tag{44}$$

[1] 2021.6.22

$$\beta_1 = \frac{13}{12}\left(u_{i-1} - 2u_i + u_{i+1}\right)^2 + \frac{1}{4}\left(u_{i-1} - u_{i+1}\right)^2 \tag{45}$$

$$\beta_3 = \frac{13}{12}\left(u_{i-2} - 2u_{i-1} + u_i\right)^2 + \frac{1}{4}\left(u_{i-2} - 4u_{i-1} + 3u_i\right)^2 \tag{46}$$

$$\alpha_0 = \frac{0.3}{(\epsilon+\beta_0)^2}, \quad \alpha_1 = \frac{0.6}{(\epsilon+\beta_1)^2}, \quad \alpha_2 = \frac{0.1}{(\epsilon+\beta_2)^2} \tag{47}$$

$$\omega_r = \frac{\alpha_r}{\sum_{j=0}^{2}\alpha_j}, \quad r = 0,1,2 \tag{48}$$

$$\widetilde{\beta}_0 = \frac{13}{12}\left(u_{i+1} - 2u_i + u_{i-1}\right)^2 + \frac{1}{4}\left(3u_{i+1} - 4u_i + u_{i-1}\right)^2 \tag{49}$$

$$\widetilde{\beta}_1 = \frac{13}{12}\left(u_{i+2} - 2u_{i+1} + u_i\right)^2 + \frac{1}{4}\left(u_{i+2} - u_i\right)^2 \tag{50}$$

$$\widetilde{\beta}_3 = \frac{13}{12}\left(u_{i+3} - 2u_{i+2} + u_{i+1}\right)^2 + \frac{1}{4}\left(u_{i+3} - 4u_{i+2} + 3u_{i+1}\right)^2 \tag{51}$$

$$\widetilde{\alpha}_0 = \frac{0.3}{(\epsilon+\widehat{\beta}_0)^2}, \quad \widetilde{\alpha}_1 = \frac{0.6}{(\epsilon+\widetilde{\beta}_1)^2}, \quad \widetilde{\alpha}_2 = \frac{0.1}{(\epsilon+\widetilde{\beta}_2)^2} \tag{52}$$

$$\widetilde{\omega}_r = \frac{\widetilde{\alpha}_r}{\sum_{j=0}^{2}\widetilde{\alpha}_j}, \quad r = 0,1,2 \tag{53}$$

合成 5 阶 WENO 近似：

$$u^-_{i+\frac{1}{2}} = \sum_{r=0}^{2}\omega_r u^{-,(r)}_{i+\frac{1}{2}}, \quad u^+_{i+\frac{1}{2}} = \sum_{r=0}^{2}\widetilde{\omega}_r u^{+,(r)}_{i+\frac{1}{2}} \tag{54}$$

3.3 空间离散—Lax Friedrichs 通量分裂

对于固定的 t，考虑逼近 $\frac{\partial}{\partial x} f\left(u(x,t)\right)$.

采用 Lax-Friedrichs 通量分裂计算数值通量：

$$f(u) = f^+(u) + f^-(u), \quad \frac{\mathrm{d}}{\mathrm{d}u}f^+(u) \geqslant 0,\ \frac{\mathrm{d}}{\mathrm{d}u}f^-(u) \leqslant 0 \tag{55}$$

$$f^{\pm}(u) = \frac{1}{2}\left(f(u) \pm \alpha u\right) \tag{56}$$

$$\alpha = \max_u \left|f'(u)\right| = \max_{x\in[a,b]}\left|f'\left(u(x,t_n)\right)\right| \approxeq \max_{1\leqslant i\leqslant N}\left|f'(u_i^n)\right| \tag{57}$$

其中，f^+ 利用偏左的模板，f^- 利用偏右的模板。

算法：Lax-Friedrichs 通量分裂计算数值通量：

1. 估计通量分裂参数 $\alpha = \max\limits_{1\leqslant i\leqslant N}|f'(u_i^n)|$，或 $\alpha = 1.01\max\limits_{1\leq i\leqslant N}|f'(u_i^n)|$.
2. 输入 $f^+(u_j),\ j = i-1, i, i+1$，使用 k 阶 WENO 左侧近似 $\widehat{f}^{+,-}_{i+\frac{1}{2}}$.
3. 输入 $f^-(u_j),\ j = i, i+1, i+2$，使用 k 阶 WENO 右侧近似 $\widehat{f}^{-,+}_{i+\frac{1}{2}}$.
4. 相加，合成数值通量 $\widehat{f}_{i+\frac{1}{2}} = \widehat{f}^{+,-}_{i+\frac{1}{2}} + \widehat{f}^{-,+}_{i+\frac{1}{2}}$.

然后，导数近似即为：

$$\frac{\partial}{\partial x}f(u(x_i,t)) = \frac{1}{\Delta x}\left(\widehat{f}_{i+\frac{1}{2}} - \widehat{f}_{i-\frac{1}{2}}\right) + \mathcal{O}\left(\Delta x^k\right) \tag{58}$$

此时，定义空间离散算子 $\mathcal{L}$ 为：

$$\mathcal{L}(u) = -\frac{1}{\Delta x}\left(\widehat{f}_{i+\frac{1}{2}} - \widehat{f}_{i-\frac{1}{2}}\right) \tag{59}$$

则单步向前 Euler 格式为：

$$u^{n+1} = u^n + \Delta t\mathcal{L}(u^n) + \Delta t g^n \tag{60}$$

3.4 时间离散

最后进行时间离散，将时间 $[0,T]$ 等距划分为 $0=t_0<\cdots<t_M=T$，网格大小 $\Delta t=\frac{T}{m}$. 采用以下 TVD Runge Kutta 算法：

- 2 阶 TVD Runge Kutta 算法，$c=1$：

$$u^{(1)}=u^n+\Delta t\mathcal{L}\left(u^n\right)+\Delta tg^n \tag{61}$$
$$u^{n+1}=\frac{1}{2}u^{(1)}+\frac{1}{2}u^{(1)}+\frac{1}{2}\Delta t\mathcal{L}\left(u^{(1)}\right)+\frac{1}{2}\Delta tg^{n+1} \tag{62}$$

- 3 阶 TVD Runge Kutta 算法，$c=1$：

$$u^{(1)}=u^n+\Delta t\mathcal{L}\left(u^n\right)+\Delta tg^n \tag{63}$$
$$u^{(2)}=\frac{3}{4}u^n+\frac{1}{4}u^{(1)}+\frac{1}{4}\Delta t\mathcal{L}\left(u^{(1)}\right)+\frac{1}{4}\Delta tg^{n+1} \tag{64}$$
$$u^{n+1}=\frac{1}{3}u^n+\frac{2}{3}u^{(2)}+\frac{2}{3}\Delta t\mathcal{L}\left(u^{(2)}\right)+\frac{2}{3}\Delta tg^{n+\frac{1}{2}} \tag{65}$$

根据 Courant-Friedrichs-Levy 条件，我们需要取：

$$\frac{\Delta t}{\Delta x}\max_{(x,t)\in[a,b]\times[0,T]}\left|f'\left(u(x,t)\right)\right|\leqslant c \tag{66}$$

注记：

- 当使用精确边界条件时，一些理论推导显示，Runge Kutta 算法的应用会引起一些收敛阶降低的现象 (同时也在本文程序结果中有体现)，这是因为在计算中间值 $u^{(1)},u^{(2)}$ 等时，其应用了边界条件，但是误差项并不符合边界项的误差。

3.5 Algorithm for the WENO procedure in 1D

Algorithm 1 1-dimensional WENO-3 for $u_t+f(u)_x=0$ on uniform grids

Input: $f(u)$, $u_0(x)$, $g(x,t)$, boundary conditions(BC), $[a,b]$, stopping time T, N, M

Output: u_i^n, $1\leqslant i\leqslant N$, $0\leqslant n\leqslant M$

1: $\Delta x\leftarrow\frac{b-a}{N}$, $\Delta t\leftarrow\frac{T}{M}$, $x_i\leftarrow a+\left(i-\frac{1}{2}\right)\Delta x$, $t_n\leftarrow n\Delta t$;
2: $u(1:N,0)\leftarrow u_0\left(x(1:N)\right)$;
3: **for** $j=0:M-1$ **do**
4: $\quad u(1:N,j+1)\leftarrow$**TVDRK**$(u(1:N,j),\mathcal{L},g(x,t),x(1:N),t,\Delta t)$;
5: **end for**
6: **return** $u(1:N,0:M)$;

Algorithm 2 Time-forward-with-TVDRK3 (TVDRK3)

Input: $u(1:N)$, $\mathcal{L}$, $g(x,t)$, $x(1:N)$, t, Δt //L is the spatial discretization operator for approximating u_t

Output: $v(1:N)$

$u^{(1)}(1:N) \leftarrow u(1:N) + \Delta t\mathcal{L}\left(u(1:N)\right) + \Delta t g(x(1:N),t)$;

$u^{(2)}(1:N) \leftarrow \frac{3}{4}u^n(1:N) + \frac{1}{4}\left(u^{(1)}(1:N) + \Delta t\left(\mathcal{L}\left(u^{(1)}(1:N)\right) + g(x(1:N), t+\Delta t)\right)\right)$;

$v(1:N) \leftarrow \frac{1}{3}u(1:N) + \frac{2}{3}\left(u^{(2)}(1:N) + \Delta t\left(\mathcal{L}\left(u^{(2)}(1:N)\right) + g(x(1:N), t+\frac{1}{2}\Delta t)\right)\right)$;

return $v(1:N)$;

Algorithm 3 Time-forward-with-TVDRK2 (TVDRK2)

Input: $u(1:N)$, $\mathcal{L}$, $g(x,t)$, $x(1:N)$, t, Δt //L is the spatial discretization operator for approximating u_t

Output: $v(1:N)$

$u^{(1)}(1:N) \leftarrow u(1:N) + \Delta t\mathcal{L}\left(u(1:N)\right) + \Delta t g(x(1:N),t)$;

$v(1:N) \leftarrow \frac{1}{2}u(1:N) + \frac{1}{2}\left(u^{(1)}(1:N) + \Delta t\left(\mathcal{L}\left(u^{(1)}(1:N)\right) + g(x(1:N), t+\Delta t)\right)\right)$;

return $v(1:N)$;

Algorithm 4 Spatial discretization operator $\mathcal{L}$ ($\mathcal{L}$)

Input: $u(1:N)$, boundary conditions(BC), Δx

Output: $v(1:N)$

$(b_{-1}, b_0, b_{N+1}, b_{N+2}) \leftarrow$**Handle-Boundary**(BC)$(u(1:N))$;

$w(-1:N+2) \leftarrow (b_{-1}, b_0, u(1:N), b_{N+1}, b_{N+2})$

$farr = f(w(-1:N+2))$

$\alpha \leftarrow \max_w |f'(w)|$;

$v_{\pm}(-1:N+2) = \frac{1}{2}\left(farr(-1:N+2) \pm \alpha(b_{-1}, b_0, u(1:N), b_{N+1}, b_{N+2})\right)$;

//Lax-Friedrichs flux splitting

$flux_{+}^{-}(0:N) \leftarrow$**Reconstruction**($v_{+}(-1:N+2)$,left); //$flux_{+}^{-}(i) = v_{+,i+\frac{1}{2}}^{-}$

$flux_{-}^{+}(0:N) \leftarrow$**Reconstruction**($v_{-}(-1:N+2)$,right); //$flux_{-}^{+}(i) = v_{-,i+\frac{1}{2}}^{+}$

$v(1:N) \leftarrow \frac{1}{\Delta x}\left(flux_{+}^{-}(0:N-1) - flux_{+}^{-}(1:N) + flux_{-}^{+}(0:N-1) - flux_{-}^{+}(1:N)\right)$;

return $v(1:N)$;

Algorithm 5 WENO3 Reconstruction v on the cell boundaries (Reconstruction)

Input: $u(-1:N+2)$, direction

Output: $v(0:N)$

if direction == left **then**
  for $i=0:N$ **do**
    $approx(1)=\frac{u(i)+u(i+1)}{2}$, $approx(2)=\frac{-u(i-1)+3u(i)}{2}$
    $\beta(1)=(u(i+1)-u(i))^2$, $\beta(2)=(u(i)-u(i-1))^2$
    $weight(1)=\frac{2/3}{(\epsilon+\beta(1))^2}$, $weight(2)=\frac{1/3}{(\epsilon+\beta(2))^2}$
    $v(i)=\frac{\mathrm{sum}(weight*approx)}{\mathrm{sum}(weight)}$
  end for
else
  for $i=0:N$ **do**
    $approx(1)=\frac{u(i)+u(i+1)}{2}$, $approx(2)=\frac{-u(i+2)+3u(i+1)}{2}$
    $\beta(1)=(u(i+1)-u(i))^2$, $\beta(2)=(u(i+1)-u(i+2))^2$
    $weight(1)=\frac{2/3}{(\epsilon+\beta(1))^2}$, $weight(2)=\frac{1/3}{(\epsilon+\beta(2))^2}$
    $v(i)=\frac{\mathrm{sum}(weight*approx)}{\mathrm{sum}(weight)}$
  end for
end if
return $v(0:N)$;

4 Multi dimensional WNEO procedure

考虑 $D\subset\mathbb{R}^2$ 具有 $D=[a,b]\times[c,d]$ 的形式，将其划分为 $N_x\times N_y$ 的等距笛卡尔网格：

$$a=x_{\frac{1}{2}}<x_{\frac{3}{2}}<\cdots<x_{N_x+\frac{1}{2}}=b,\quad c=y_{\frac{1}{2}}<y_{\frac{3}{2}}<\cdots<y_{N_y+\frac{1}{2}}=d \tag{67}$$

$$(x_i,y_j)=\left(\frac{x_{i-\frac{1}{2}}+x_{i+\frac{1}{2}}}{2},\frac{y_{i-\frac{1}{2}}+y_{i+\frac{1}{2}}}{2}\right)=\frac{1}{4}\sum_{2^2}\left(x_{i\pm\frac{1}{2}},y_{j\pm\frac{1}{2}}\right) \tag{68}$$

$$\Delta x=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}},\quad \Delta y=y_{j+\frac{1}{2}}-y_{j-\frac{1}{2}},\quad \Delta=\max\{\Delta x,\Delta y\} \tag{69}$$

4.1 Spatial discretization

我们将上述方程离散为：

$$\frac{\mathrm{d}}{\mathrm{d}t}u_{i,j}=-\frac{\widehat{f}_{i+\frac{1}{2},j}-\widehat{f}_{i-\frac{1}{2},j}}{\Delta x}-\frac{\widehat{g}_{i,j+\frac{1}{2}}-\widehat{g}_{i,j-\frac{1}{2}}}{\Delta y} \tag{70}$$

其中 $\widehat{f},\widehat{g}$ 是数值通量，使得上面对于 $\frac{\partial}{\partial x}f(u(x,y,t)),\frac{\partial}{\partial y}f(u(x,y,t))$ 的近似具有 k 阶误差阶。

由上式的形式，我们只需要在每一个 (x_i,y_j) 处如同一维情形，分别重构 $\widehat{f},\widehat{g}$ 即可。算法如下：

1. 处理边界，加入虚拟点。
2. 对于每一对 (i,j)，重构 $\widehat{f}_{i\pm\frac{1}{2},j}$, $\widehat{g}_{i,j\pm\frac{1}{2}}$ (需要使用通量分裂，具体过程和一维情形完全相同).
3. 计算 x, y 方向的偏导数，取相反数相加，合成空间离散算子 $\mathcal{L}$.

5 Characteristic-wise WENO procedure for systems

考虑双曲守恒律问题：

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{0} \tag{71}$$

其中 $\boldsymbol{u} \in \mathbb{R}^m$，$\boldsymbol{f} : \mathbb{R}^m \mapsto \mathbb{R}^m$.

因为上述是一个双曲系统，所以 f 的 Jacobi 矩阵 $\boldsymbol{f}' \in \mathbb{R}^{m\times m}$ 是可以实数对角化，我们设其右特征方阵为方阵 $\boldsymbol{R}(u) \in \mathrm{GL}_m(\mathbb{R})$，对应的特征值对角方阵为 $\boldsymbol{\Lambda}(u)$，也即：

$$\boldsymbol{R}^{-1}(\boldsymbol{u})\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{\Lambda}(\boldsymbol{u}) = \mathrm{diag}\left(\lambda_1(\boldsymbol{u}), \cdots, \lambda_m(\boldsymbol{u})\right) \tag{72}$$

我们记 $\boldsymbol{R}(u)$ 的各列，也即 Jacobi 阵 $\boldsymbol{f}'$ 的右特征向量为：

$$\boldsymbol{R}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{r}_1(u) & \boldsymbol{r}_2(u) & \cdots & \boldsymbol{r}_m(u) \end{pmatrix} \tag{73}$$

同时记 $\boldsymbol{R}^{-1}(u)$ 的各行，也即 Jacobi 阵 $\boldsymbol{f}'$ 的左特征向量为：

$$\boldsymbol{R}^{-1}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{l}_1(\boldsymbol{u}) \\ \boldsymbol{l}_2(\boldsymbol{u}) \\ \vdots \\ \boldsymbol{l}_m(\boldsymbol{u}) \end{pmatrix} \tag{74}$$

5.1 Spatial discretization

对于每一个给定的 i, $i = 0, \cdots, N$，我们用如下方式计算数值通量 $\widehat{\boldsymbol{f}}_{i+\frac{1}{2}}$ 的重构。

首先计算 $\boldsymbol{f}'$ 在 $x_{i+\frac{1}{2}}$ 处的 (近似) 取值，我们可以使用：

- 均值：$\boldsymbol{f}'(\boldsymbol{u}_{i+\frac{1}{2}}) \approx \boldsymbol{f}'(\frac{\boldsymbol{u}_i+\boldsymbol{u}_{i+1}}{2})$.(矩阵有显式表达式时，较为方便)
- Roe 均值：$\boldsymbol{f}(\boldsymbol{u}_{i+1}) - \boldsymbol{f}(\boldsymbol{u}_i) = \boldsymbol{f}'(\boldsymbol{u}_{i+\frac{1}{2}})(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i)$.

然后计算 $\boldsymbol{f}'(\boldsymbol{u}_{i+\frac{1}{2}})$ 的特征分解：

$$\boldsymbol{R}^{-1}_{i+\frac{1}{2}}\boldsymbol{f}'_{i+\frac{1}{2}}\boldsymbol{R}_{i+\frac{1}{2}} = \boldsymbol{\Lambda}_{i+\frac{1}{2}} = \mathrm{diag}\left(\lambda^{(1)}_{i+\frac{1}{2}}, \cdots, \lambda^{(m)}_{i+\frac{1}{2}}\right) \tag{75}$$

在 $x_{i+\frac{1}{2}}$ 附近将方程写成非守恒形式：

$$\boldsymbol{u}_t + \boldsymbol{f}'(\boldsymbol{u})\boldsymbol{u}_x = \boldsymbol{0} \iff \boldsymbol{u}_t + \boldsymbol{R}_{i+\frac{1}{2}}\left(\boldsymbol{R}^{-1}_{i+\frac{1}{2}}\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{R}_{i+\frac{1}{2}}\right)\boldsymbol{R}^{-1}_{i+\frac{1}{2}}\boldsymbol{u}_x = \boldsymbol{0} \tag{76}$$

代换 $\boldsymbol{g}(\boldsymbol{v}) = \boldsymbol{R}^{-1}_{i+\frac{1}{2}}\boldsymbol{f}(\boldsymbol{R}_{i+\frac{1}{2}}\boldsymbol{v})$, $\boldsymbol{v} = \boldsymbol{R}^{-1}_{i+\frac{1}{2}}\boldsymbol{u}$，那么方程等价为：

$$\boldsymbol{R}_{i+\frac{1}{2}}\boldsymbol{v}_t + \boldsymbol{R}_{i+\frac{1}{2}}\boldsymbol{g}'(\boldsymbol{v})\boldsymbol{v}_x = \boldsymbol{0} \iff \boldsymbol{v}_t + \boldsymbol{g}(\boldsymbol{v})_x = \boldsymbol{0} \tag{77}$$

其中 $\boldsymbol{g}'(\boldsymbol{v}) = \boldsymbol{R}^{-1}_{i+\frac{1}{2}}\boldsymbol{f}'(\boldsymbol{R}_{i+\frac{1}{2}}\boldsymbol{v})\boldsymbol{R}_{i+\frac{1}{2}} \approx \boldsymbol{\Lambda}_{i+\frac{1}{2}}$. 那么方程**近似**为：

$$\boldsymbol{v}_t + \boldsymbol{\Lambda}_{i+\frac{1}{2}}\boldsymbol{v}_x = \boldsymbol{0} \iff \boldsymbol{v}^{(l)}_t + \boldsymbol{g}^{(l)}_x = 0, \quad 1 \leqslant l \leqslant m \tag{78}$$

使用 Lax-Friedrichs 通量分裂，第 l 个分量的系数 $\alpha_l = \max_{\boldsymbol{u}} \left|\lambda^{(l)}(\boldsymbol{u})\right|$.

数值计算时，记 $\boldsymbol{g}_j=\boldsymbol{R}_{i+\frac{1}{2}}^{-1}\boldsymbol{f}(\boldsymbol{u}_j)$, $\boldsymbol{v}_j=\boldsymbol{R}_{i+\frac{1}{2}}^{-1}\boldsymbol{u}_j$，取 $\alpha_l=\max\limits_{1\leqslant j\leqslant N}\left|\lambda_{j+\frac{1}{2}}^{(l)}\right|$，然后逐分量重构 $\widehat{\boldsymbol{g}}_{i+\frac{1}{2}}$，于是 $\widehat{\boldsymbol{f}}_{i+\frac{1}{2}}=\boldsymbol{R}_{i+\frac{1}{2}}\widehat{\boldsymbol{g}}_{i+\frac{1}{2}}$.

第三部分　Hamilton-Jacobi Equations

本部分，我们考虑应用 WENO 方法求解 Hamilton-Jacobi 方程：

$$\begin{cases}\frac{\partial}{\partial t}\varphi+H\left(\nabla_{\boldsymbol{x}}\varphi\right)=0\\ \varphi(\boldsymbol{x},0)=\varphi_0(\boldsymbol{x})\end{cases} \tag{79}$$

其中 $H(\cdot)$ 是 Lipshitz 连续的函数。

注记：Hamilton-Jacobi 方程不能使用 FVM；而且，使用 WENO 求解时是对待求解函数的导数进行 FDM 逼近，且只需要正交网格，不一定需要光滑网格。

6　Spatial Discretization

我们采用 monotone Hamiltonian 进行计算。其中，monotone Hamiltonian 为 $\widehat{H}\left(\cdot,\cdot\right)$ 满足：

- 单调性：$\widehat{H}\left(\cdot,\cdot\right)=\widehat{H}\left(\uparrow,\downarrow\right)$，其中单调性是对每个分量而言的。
- 相容性：$\widehat{H}\left(u,u\right)=H\left(u\right)$.
- Lipshitz 连续性：对每个变量都 Lipschitz 连续。

例如，可以选取 Godunov monotone Hamiltonian：

$$\widehat{H}\left(u^-,u^+\right)=\mathrm{ext}_{u\in I(u^-,u^+)}H(u) \tag{80}$$

$$\mathrm{ext}_{u\in I(a,b)}=\begin{cases}\min_{u\in(a,b)}, & a<b\\ \max_{u\in(b,a)}, & a\geqslant b\end{cases} \tag{81}$$

也可以使用 Lax-Friedrichs 分裂：

$$\widehat{H}\left(u^-,u^+\right)=H\left(\frac{u^-+u^+}{2}\right)-\frac{\alpha}{2}\left(u^+-u^-\right) \tag{82}$$

$$\alpha=\max_{u\in I[u^-,u^+]}\left|H'(u)\right| \tag{83}$$

那么，我们需要对每一个分量，在 $(x_{1i},x_{2i},\cdots,x_{mi})$ 处重构 $\frac{\partial}{\partial x_k}\varphi$. 我们下面以一维 WENO5 为例，给出具体的实现方法。

由于 H 是关于 $\nabla_{\boldsymbol{x}}\varphi$ 的函数，所以我们需要在 x_i 处重构 $u_i^{\pm}$，其中 $u=\frac{\partial}{\partial x}\varphi$.

我们记：

$$\Delta^-\varphi_i=\frac{\varphi_i-\varphi_{i-1}}{\Delta x} \tag{84}$$

$$\Delta^+\varphi_+=\frac{\varphi_{i+1}-\varphi_i}{\Delta x} \tag{85}$$

于是：

$$\frac{1}{\Delta x}\int_{x_{i-1}}^{x_i}u(\xi)\mathrm{d}\xi=\Delta^-\varphi_i \tag{86}$$

$$\frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} u(\xi)\mathrm{d}\xi = \Delta^+\varphi_i \tag{87}$$

那么，重构 $u_i^\pm$ 只需要利用 $\Delta^\pm\varphi_j$ 即。具体地，对于 WENO5，我们只需用 $\Delta^-\varphi_j$ 替代 u_j 输入，其中 $j = i-2,\cdots,i+2$，即可得到 u_i^-. 对称地，重构 u_i^+ 时，需要 $\Delta^-\varphi_j$, $j = i-1,\cdots,i+3$. 因此，大模板包含 3 个小模版，总共涉及 6 个点。

第四部分　Numerical Examples

7　Examples for General Hyperbolic Conservation Law

7.1　Example 1—A Simple 1D Linear Case

考虑问题：

$$\begin{cases} u_t + u_x = 0, & t\in[0,1],\ x\in[0,1] \\ u(x,0) = \sin(2\pi x), & x\in[0,1] \\ u(0,t) = u(1,t), & t\in[0,1] \end{cases} \tag{88}$$

易知其真解为：

$$u(x,t) = \sin\left(2\pi(x-t)\right) \tag{89}$$

注记：我们实验时，u 在 $[0,1]$ 外的点取值按照周期条件处理。

使用 TVDRK2/3 做时间离散，3 阶 WENO 做空间离散，选取网格加密方式：

$$N_k = 2^{k+3},\quad M_k = \frac{5}{4}N_k,\quad k = 1,\cdots$$

此时，$\frac{\Delta t}{\Delta x} = 0.8$.[2]

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.27299579947968716	0.0000000000000000	0.37640745041025647	0.0000000000000000
32	0.10812570032129067	1.3361692759193367	0.16638756084179451	1.1777476028729192
64	4.2210480337538353E-002	1.3570363252317466	7.5292054706546929E-002	1.1439780456761424
128	1.2399754218359477E-002	1.7672897221266535	2.8520650494934463E-002	1.4004907436909098
256	1.2045653158317584E-003	3.3637269956404285	4.3484429600030294E-003	2.7134360719016253
512	1.2381656356660268E-004	3.2822363946011026	5.1789593046169813E-004	3.0697647827525820
1024	1.8521292371395900E-005	2.7409476490335010	2.5266216744132541E-005	4.3573806595150106
2048	4.4661156907154966E-006	2.0520922414367888	6.3100409728560070E-006	2.0014883792607709
4096	1.1151898445393380E-006	2.0017312948669801	1.5771293141106368E-006	2.0003484166866228
8192	2.7878774065376595E-007	2.0000503015137028	3.9426639855092045E-007	2.0000582896088890
16384	6.9696894383645579E-008	2.0000008441235813	9.8566304496268929E-008	2.0000043199193382
32768	1.7424224382136717E-008	1.9999999349019288	2.4641574914531289E-008	2.0000000708149326

表 2: results for WENO3 with TVDRK2, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$

[2] 2021.6.19

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.16850281978711509	0.0000000s000000000	0.27705174830915191	0.0000000000000000
32	6.6580589627490833E-002	1.3395991822562039	0.11620244546363667	1.2535150406922588
64	2.1770054004877026E-002	1.6127566604029884	4.7864368893342002E-002	1.2796164381785553
128	5.6322557287645653E-003	1.9505602412495748	1.6451007440321352E-002	1.5407761514678695
256	1.0530542076939210E-003	2.4191331366850197	4.0890284478339867E-003	2.0083459306064544
512	9.9066833097257524E-005	3.4100337593974541	5.1452664171858231E-004	2.9904404142972654
1024	4.9443906977129113E-006	4.3245374814724604	2.3988046342404701E-005	4.4228582103139944
2048	2.2232650934861665E-007	4.4750408620810092	7.7781798235587729E-007	4.9467392660393275
4096	9.9522708726051073E-009	4.4815104176709903	2.6092995653037576E-008	4.8976980988296468
8192	4.6917344982718719E-010	4.4068324723116241	1.0385429183301653E-009	4.6510298421930489
16384	3.3124400611704863E-011	3.8241551204019899	6.1454508148983678E-011	4.0788981803384958
32768	5.1738982689928316E-012	2.6785707564969790	7.6941786275597224E-012	2.9976796183271825

表 3: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$

我们发现，上述 4 个结果都具有收敛阶先增加后减少的趋势，而且都先升高至高于理论值的收敛阶。这似乎与 WENO 权重的取值有关。作为验证，我们下面进行了几个实验：[3]

首先，我们将 WENO 权重变为固定值，计算了 2 个例子，发现收敛阶很理想，这验证了 Runge Kutta 格式应用的正确性。

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	6.7501127616801410E-002	0.0000000000000000	9.5265396299910765E-002	0.0000000000000000
32	1.7921263000490080E-002	1.9132392879395150	2.5314922775265258E-002	1.9119641891248549
64	4.5459503583751074E-003	1.9790184796964418	6.4264461522436436E-003	1.9778950355707907
128	1.1405635232760326E-003	1.9948351307903591	1.6128233839522062E-003	1.9944326811378001
256	2.8539431630927067E-004	1.9987182906475325	4.0359680006750789E-004	1.9986018202628586
512	7.1364370292039425E-005	1.9996807310428053	1.0092370815940541E-004	1.9996496158322121
1024	1.7842077889972947E-005	1.9999203259195710	2.5232460896484205E-005	1.9999122973130095
2048	4.4605810018531292E-006	1.9999800992932111	6.3082111526661065E-006	1.9999780608661981
4096	1.1151490942874272E-006	1.9999950271442468	1.5770587860398401E-006	1.9999945131287327
8192	2.7878751364415638E-007	1.9999987576514435	3.9426507173736993E-007	1.9999986269669321
16384	6.9696893381854736E-008	1.9999996901106671	9.8566291450091293E-008	1.9999996558046595

表 4: results for WENO3 with TVDRK2, periodic boundary, fixed weights, $\frac{\Delta t}{\Delta x} = 0.8$

[3] 2021.6.22

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	2.6979274172364297E-002	0.0000000000000000	3.7880957754901856E-002	0.0000000000000000
32	3.4902816189739750E-003	2.9504361833490007	4.9268787014965509E-003	2.9427269489174983
64	4.3921396755229418E-004	2.9903476076081241	6.2085219661356383E-004	2.9883521945459672
128	5.4979439427140397E-005	2.9979598323209995	7.7743561376686365E-005	2.9974547505161375
256	6.8746311612402449E-006	2.9995379801224935	9.7219116960145513E-006	2.9994112626939775
512	8.5939410364296119E-007	2.9998905280696513	1.2153578876628046E-006	2.9998588219502555
1024	1.0742629451337337E-007	2.9999727166762691	1.5192344027248339E-007	2.9999648226666320
2048	1.3428450570051288E-008	2.9999824066645133	1.8990680028529994E-008	2.9999810081495295
4096	1.6787637048160515E-009	2.9998217678200993	2.3741137944455204E-009	2.9998305751655958
8192	2.1025684558128428E-010	2.9971744994483429	2.9733948636589957E-010	2.9972061174978637
16384	2.7104062470397434E-011	2.9555717627036455	3.8217429221276689E-011	2.9598084236634428

表 5: results for WENO3 with TVDRK3, periodic boundary, fixed weights, $\frac{\Delta t}{\Delta x} = 0.8$

然后，考虑到数值格式应保持某种”self similar” 的性质，我们取 WENO 权重中 $\epsilon = \Delta x^2$. 虽然同样具有收敛阶由低增长到高于理论值，然后再逼近理论值的过程，但是明显比之前的结果理想。实验结果如下：

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.24775289683007723	0.0000000000000000	0.34173553359795550	0.0000000000000000
32	8.2160222502007482E-002	1.5923899322966741	0.13586984280444261	1.3306549906451548
64	1.3244220049231798E-002	2.6330772026472409	2.6956529156051623E-002	2.3335186196956319
128	2.4693169106270287E-003	2.4231789779674839	6.4325421371810299E-003	2.0671738436194338
256	4.2254394946487862E-004	2.5469386916589491	1.2030438052114123E-003	2.4186998276877096
512	8.2217515976654241E-005	2.3615837164120554	1.8317584861082814E-004	2.7153879703848123
1024	1.8569639373387799E-005	2.1464999865886583	2.5266290975767269E-005	2.8579434982129910
2048	4.5068145928289202E-006	2.0427657931490755	6.3103233131876768E-006	2.0014280664785327
4096	1.1180501881222337E-006	2.0111231490882422	1.5771907742358204E-006	2.0003567478939153
8192	2.7896900898594176E-007	2.0028081854472757	3.9427332113387004E-007	2.0000891789735120
16384	6.9708239769104604E-008	2.0007037570496808	9.8566807228781667E-008	2.0000222923571114

表 6: results for WENO3 with TVDRK2, periodic boundary, $\epsilon = \Delta x^2$, $\frac{\Delta t}{\Delta x} = 0.8$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.14928021869205679	0.0000000000000000	0.25285113298228912	0.0000000000000000
32	4.7740936623399108E-002	1.6447242289570529	8.6008967179207585E-002	1.5557292567342431
64	1.0956621730502670E-002	2.1234238319347845	2.5537371411478604E-002	1.7518770465368643
128	1.9993480655217319E-003	2.4542014816840791	6.1598585707001163E-003	2.0516409021818993
256	3.0164560392692509E-004	2.7286031896987928	1.1884635179563174E-003	2.3737996094503182
512	4.0316099028994177E-005	2.9034266011472285	1.8575700291356689E-004	2.6776091122880517
1024	5.1202362824750091E-006	2.9770737583251159	2.5107389183931517E-005	2.8872326789310088
2048	6.4174565151826604E-007	2.9961368672931963	3.1927075370141722E-006	2.9752597081262371
4096	8.0250392126348834E-008	2.9994212683174983	4.0014564706147837E-007	2.9961832829036847
8192	1.0031958955419817E-008	2.9999050910248246	5.0036801813035936E-008	2.9994637298537064
16384	1.2540620226078776E-009	2.9999227437298774	6.2560220337459782E-009	2.9996720812055235

表 7: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = \Delta x^2$, $\frac{\Delta t}{\Delta x} = 0.8$

使用 TVDRK3 做时间离散，5 阶 WENO 做空间离散，选取网格加密方式：[4]

$$N_k = 2^{k+3}, M = \left[N^{\frac{5}{3}}\right], k = 1, \cdots$$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	7.1854924773114698E-004	0.0000000000000000	1.0146106418015277E-003	0.0000000000000000
32	2.2851343038901571E-005	4.9747361748992267	3.2304431964669256E-005	4.9730502800511207
64	7.1704669971750305E-007	4.9940680672423676	1.0139658426222553E-006	4.9936511488416890
128	2.2433576817888649E-008	4.9983355139146317	3.1725161830920001E-008	4.9982377196820922
256	7.0164218097393896E-010	4.9987803707668164	9.9223429472772295E-010	4.9988028909312465
512	2.3310830746615501E-011	4.9116631566429563	3.2763902702015457E-011	4.9205017018987265

表 8: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta t \approx \Delta x^{\frac{5}{3}}$

易见，收敛阶于理论值相接近，则空间离散误差正确。

同时，我们选取 $\frac{\Delta t}{\Delta x} = 0.8$，以验证时间离散正确性[5]。易见，收敛阶符合时间离散的收敛阶 3 阶，也即时间离散误差不可忽略：

[4] 2021.6.19

[5] 2021.6.22

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	9.5935247352502331E-003	0.0000000000000000	1.4314040113959647E-002	0.0000000000000000
32	8.5721663101213496E-004	3.4843292245852338	1.2011262302610293E-003	3.5749712461745919
64	9.4281710296676240E-005	3.1846100054378574	1.3211808541602199E-004	3.1844879032814846
128	1.1356971641738082E-005	3.0533997409932252	1.6007439115672817E-005	3.0450135403368650
256	1.4059262623393730E-006	3.0139853533178678	1.9859665044696939E-006	3.0108293271499806
512	1.7530988845043313E-007	3.0035416507840789	2.4778085350174450E-007	3.0027046733118952
1024	2.1900309459529767E-008	3.0008842134964127	3.0966498920648178E-008	3.0002845260745903
2048	2.7372233472664928E-009	3.0001661926092251	3.8708662941644434E-009	2.9999799001677445
4096	3.4234593539952146E-010	2.9991863692339078	4.8413151265691567E-010	2.9991855665380029
8192	4.3203999589472891E-011	2.9862181032374049	6.1072258361605236E-011	2.9868099038701268

表 9: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$

7.2 Example 2—1D Burgers Equation with a Source

考虑问题：

$$\begin{cases} u_t + \left(\frac{u}{2}\right)^2_x = \mathrm{e}^t \sin(2\pi x)\left(1 + 2\pi \mathrm{e}^t \cos(2\pi x)\right), & t \in [0,1],\, x \in [0,1] \\ u(x,0) = \sin(2\pi x), & x \in [0,1] \\ u(0,t) = u(1,t) = 0, & t \in [0,1] \end{cases} \tag{90}$$

易知其真解为：

$$u(x,t) = \mathrm{e}^t \sin(2\pi x) \tag{91}$$

注记：我们实验时，u 在 $[0,1]$ 外的点取值直接取真解的值 (Dirichlet boundary condition)。易知 $\max\limits_u |f'(u)| = \mathrm{e}$. 选取 $M = 4N$，选取网格加密方式：

$$N_k = 2^{k+3}, \quad M_k = 4N_k, \quad k = 1, \cdots$$

此时，$\frac{\Delta t}{\Delta x} \max\limits_{[0,1]\times[0,1]} |f'(u)| = \frac{\mathrm{e}}{4} \approx 0.6769$. 选取每次的 α 估计：$\alpha = 1.01 \max\limits_{1\leqslant i\leqslant N} |f'(u_i^n)|$.[6]

[6] 2021.6.19

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.45629139792942897	0.0000000000000000	1.0568776629105909	0.0000000000000000
32	8.3151195636607472E-002	2.4561465427641895	0.26896931158442472	1.9742949086044039
64	7.5463977006710896E-003	3.4618769699235958	2.8113268982845330E-002	3.2581184522158209
128	1.0991662572945641E-003	2.7793785117937788	4.5664153282210240E-003	2.6221172308269125
256	1.7542817668818832E-004	2.6474572290718585	8.4179111814972063E-004	2.4395278898671844
512	2.5812494398727933E-005	2.7647390186736129	1.6716125380492053E-004	2.3322218028409041
1024	3.4539201517360886E-006	2.9017629275531038	3.3197984186543461E-005	2.3320729373336406
2048	5.0384742108517013E-007	2.7771759137324743	6.6165084633904026E-006	2.3269536307413490
4096	6.9131827440187148E-008	2.8655649442986988	1.1658542402870342E-006	2.5046826781835265
8192	8.1269717468163148E-009	3.0885602786526980	1.7238426493204617E-007	2.7576874306169001
16384	9.3135627149365421E-010	3.1253128265124088	2.6226628865172463E-008	2.7165238100423035

表 10: results for WENO3 with TVDRK2, Dirichlet Boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.25$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.45620138071299876	0.0000000000000000	1.0568125744481485	0.0000000000000000
32	8.3117412333242852E-002	2.4564481679259544	0.26883443255638706	1.9749297009121625
64	7.5448694395700445E-003	3.4615828990773414	2.8097812843004666E-002	3.2581881922781752
128	1.0989524507167068E-003	2.7793669704466488	4.5626824604192984E-003	2.6225036761163563
256	1.7533931870048175E-004	2.6479075129024463	8.4044149190193318E-004	2.4406629594685643
512	2.5775934241895210E-005	2.7660529235921203	1.6657794964984596E-004	2.3349499483293319
1024	3.4373837919909139E-006	2.9066418725268801	3.2960635144102524E-005	2.3373814917878590
2048	4.9765167420290880E-007	2.7881027388322828	6.5130779911895531E-006	2.3393326356845634
4096	6.6979850513868241E-008	2.8933372357121616	1.1216142363039416E-006	2.5377629361430372
8192	7.4035074007029941E-009	3.1774463437815981	1.5333654038510169E-007	2.8708031262479916
16384	6.9805666938167083E-010	3.4067928406924843	1.8320851708983343E-008	3.0651430560601085

表 11: results for WENO3 with TVDRK3, Dirichlet boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.25$

由结果可见，实验结果并不好，可能是因为边界条件使用了真解，也可能是因为具有源项。于是，我们再使用周期边界条件进行一系列实验。[7]

首先，只更改边界条件，我们有如下 4 个结果。我们发现，时间离散误差影响相对很小，而且 L^∞ 范数误差都很不理想。通过对结果的分析，发现最大模误差都出现在 $x = 0.5$ 附近.

[7] 2021.6.22

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.45538313034960831	0.0000000000000000	1.0547657958482810	0.0000000000000000
32	8.2193485000551500E-002	2.4699848983751331	0.26548157323807431	1.9902390592154111
64	7.4583980014460158E-003	3.4620863537834601	2.7685472332128369E-002	3.2614106882436387
128	1.0910060492145570E-003	2.7732066846185033	4.5112596738058619E-003	2.6175268984829745
256	1.7439397309063613E-004	2.6452370131212217	8.3427480437520130E-004	2.4349357509195193
512	2.5695584833892893E-005	2.7627577891441448	1.6554288767421654E-004	2.3333176471348755
1024	3.4343016181991536E-006	2.9034318334036402	3.2921161779747576E-005	2.3301178745037756
2048	5.0079188727152831E-007	2.7777336536721564	6.5628634745781844E-006	2.3266179224704691
4096	6.8859289356093853E-008	2.8624879939401366	1.1602744878908000E-006	2.4998592751514814

表 12: results for WENO3 with TVDRK2, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.25$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.45538313034960831	0.0000000000000000	1.0547657958482810	0.0000000000000000
32	8.2193485000551500E-002	2.4699848983751331	0.26548157323807431	1.9902390592154111
64	7.4583980014460158E-003	3.4620863537834601	2.7685472332128369E-002	3.2614106882436387
128	1.0910060492145570E-003	2.7732066846185033	4.5112596738058619E-003	2.6175268984829745
256	1.7439397309063613E-004	2.6452370131212217	8.3427480437520130E-004	2.4349357509195193
512	2.5695584833892893E-005	2.7627577891441448	1.6554288767421654E-004	2.3333176471348755
1024	3.4343016181991536E-006	2.9034318334036402	3.2921161779747576E-005	2.3301178745037756
2048	5.0079188727152831E-007	2.7777336536721564	6.5628634745781844E-006	2.3266179224704691
4096	6.8859289356093853E-008	2.8624879939401366	1.1602744878908000E-006	2.4998592751514814

表 13: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.25$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	2.1864705580037547E-002	0.0000000000000000	3.3982359898049763E-002	0.0000000000000000
32	8.3797801121912683E-004	4.7055477239698709	1.8572247668089403E-003	4.1935657135106004
64	3.0391451134319521E-005	4.7851749201164644	7.7785737725988291E-005	4.5774989582956405
128	2.2017563852444002E-006	3.7869388075429695	7.7425439004019658E-006	3.3286260933698357
256	5.0218227274969329E-007	2.1323718429440053	2.2065680064520254E-006	1.8110034461598827
512	1.3626600710158642E-007	1.8817853892234389	9.0373523233683306E-007	1.2878321391501364
1024	3.9110902303084939E-008	1.8007829874497034	3.8548652309469689E-007	1.2292197443206951
2048	1.1825265383752613E-008	1.7256982589367849	1.8331300492835378E-007	1.0723712834602377
4096	3.7698803184993442E-009	1.6492819325953707	8.6478787906726917E-008	1.0838909324301844

表 14: results for WENO5 with TVDRK2, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.25$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	2.1840702958432122E-002	0.0000000000000000	3.3850804835517279E-002	0.0000000000000000
32	8.2936250264489125E-004	4.7188726597258022	1.8183481783911049E-003	4.2184897564724801
64	2.8142986739968962E-005	4.8811554700303885	6.9434103864063346E-005	4.7108402202170332
128	9.0513422116117054E-007	4.9584998924368779	3.3488613309096138E-006	4.3739019001271036
256	3.0400073874967552E-008	4.8959850087393963	1.7222604621613957E-007	4.2812953913410947
512	9.5896903726510609E-010	4.9864467843831086	8.9762806215754054E-009	4.2620417527249659
1024	4.5809648855887604E-011	4.3877608256068044	5.2049280163668499E-010	4.1081676625548953
2048	6.0259840589405413E-012	2.9263827426630971	7.5765893559065489E-011	2.7802577489409579
4096	5.1375441848347263E-012	0.23011795987687592	1.2622771924930554E-010	-0.73640829445834921

表 15: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.25$

7.3 Example 3—1D Burgers Equation without Source

考虑 $[0,2\pi]$ 区间上周期边界的 Burgers 方程：[8]

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, & t \geqslant 0,\ x \in [0,2\pi] \\ u(x,0) = u_0(x), & x \in [0,2\pi] \\ u(0,t) = u(2\pi,t), & t \geqslant 0 \end{cases} \tag{92}$$

因为 x_0 点处的初值对应特征线为 $x = x_0 + tu_0(x_0)$，因此当特征线不相交时有：

$$u\left(x_0 + tu_0(x_0), t\right) = u_0(x_0) \tag{93}$$

对于给定的 (x,t)，则 $u = u(x,t)$ 满足的两个方程为：

$$u(x,t) = u_0(x - u(x,t)t) \tag{94}$$

$$x = x_0 + tu_0(x_0) \tag{95}$$

我们首先考虑 $u_0(x) = \sin x$ 的情形。易见其具有对称性，且间断只会出现在 $x = \pi$ 上。考虑到真解以 2π 为周期且在 $[0,2\pi]$ 上关于 $x = \pi$ 反对称，所以我们只需求出 $x \in (0,\pi)$ 上的解即可。边界：$u(0,t) = u(\pi,t) = 0$, $u(2\pi - x,t) = -u(x,t)$.

对于 x_0 发出的特征线 $x = x_0 + t\sin x_0$，我们有以下几个观察：

- $x_0 \in \left(0,\frac{\pi}{2}\right)$ 发出的特征线覆盖了缺角区域 $\left\{\min\left\{0, t-\frac{\pi}{2}\right\} < x < \pi, t > 0\right\}$，且在这个区域里两两不交。
- $x_0 = \frac{\pi}{2}$ 发出的特征线为 $x = \frac{\pi}{2} + t$.
- $x_0 \in \left(\frac{\pi}{2},\pi\right)$ 发出的特征线覆盖了三角区域 $\left\{0 < t < x - \frac{\pi}{2} < \frac{\pi}{2}\right\}$，且在这个区域里两两不交。
- $x_0 \in (0,\pi)$ 发出的特征线与 $x = \pi$ 相交于 $\varphi(x_0) = \frac{\pi - x_0}{\sin x_0}$. φ 在 $(0,\pi)$ 严格递减，取遍 $(1,+\infty)$. 这也说明了 $t = 1$ 是第一次出现激波的时刻。

同时，当 $(x,t) \in (0,\pi) \times (0,+\infty)$ 时，我们给出一些估计：

[8] 2021.6.23

- 当 $x < t + \frac{\pi}{2}$ 时，$x_0 \in \left(\max\{0, x-t\}, \min\left\{\frac{\pi}{2}, x\right\}\right)$.
- 当 $x > t + \frac{\pi}{2}$ 且 $t \leqslant 1$ 时，$x_0 \in (x-t, x)$.
- 当 $x > t + \frac{\pi}{2}$ 且 $t > 1$ 时，$x_0 \in \left(x-t, \min\left\{x, \arccos\frac{-1}{t}\right\}\right)$.

接下来，我们给出一个求 $u(x,t)$ 的算法 ($u_0(x) = \sin x$)，其中 $(x,t) \in (0,\pi) \times (0,+\infty)$.

Algorithm 6 Solution of Burgers equation on $(0, 2\pi)$ with initial value $u_0(x) = \sin x$ and periodic boundary

Input: (x, t) $(t \geqslant 0)$

Output: $u(x, t)$

if $t = 0$ **then**
  return $u = \sin x$
else if $x \notin [0, 2\pi]$ **then**
  $x \leftarrow x - \left\lfloor \frac{x}{2\pi} \right\rfloor \times 2\pi$
end if
if $x = 0, \pi, 2\pi$ **then**
  return $u = 0$
else if $x > \pi$ **then**
  $x \leftarrow 2\pi - x$
  $\mathrm{coef} \leftarrow -1$
else
  $\mathrm{coef} \leftarrow 1$
end if
if $x < t + \frac{\pi}{2}$ **then**
  $\mathrm{low} \leftarrow \max\{0, x - t\}$, $\mathrm{high} \leftarrow \min\left\{\frac{\pi}{2}, x\right\}$
else if $x = t + \frac{\pi}{2}$ **then**
  return $u = 1$
else if $t \leqslant 1$ **then**
  $\mathrm{low} \leftarrow x - t$, $\mathrm{high} \leftarrow x$
else
  $\mathrm{low} \leftarrow x - t$, $\mathrm{high} \leftarrow \min\left\{x, \arccos \frac{-1}{t}\right\}$
end if
for $i = 1$:BisecNum **do**
  $x_0 \leftarrow \frac{\mathrm{high}+\mathrm{low}}{2}$
  if $x_0 + t \sin x_0 > x$ **then**
    $\mathrm{high} \leftarrow x_0$
  else
    $\mathrm{low} \leftarrow x_0$
  end if
end for
$x_0 \leftarrow \frac{\mathrm{high}+\mathrm{low}}{2}$
$u \leftarrow$ **IterativeSolver**(x_0)
return $\mathrm{coef} \times u$

Algorithm 7 Iterative Solver

Input: x_0

Output: u

1: $u \leftarrow \sin x_0$
2: **while** $|u - \sin(x - tu)| \geqslant \epsilon = 10^{-16}$ **do**
3: $\quad u \leftarrow \frac{tu\cos(x-tu)+\sin(x-tu)}{1+t\cos(x-tu)}$
4: **end while**
5: **return** u

我们再考虑初值 $u_0(x) = a + \sin(x)$ 的情形. 使用待定系数法，令 $u(x,t) = a + v(\alpha x + \beta t, t)$，于是：

$$0 = u_t + uu_x = \beta v_x + v_t + (a+v)\alpha v_x = v_t + \alpha v v_x + (a\alpha + \beta)v_x$$

因此我们令 $\alpha = 1, \beta = -a$，$u(x,t) = a + v(x - at, t)$，$v(x,t) = u(x + at, t) - a$，就有：

$$0 = (u_t + uu_x)\big|_{(x,t)} = (v_t + vv_x)\big|_{(x-at,t)}$$

那么转化为：

$$\begin{cases} u(x,t) = v(x-at,t) + a \\ v_t + \left(\frac{v^2}{2}\right)_x = 0, & t \geqslant 0,\ x \in [0, 2\pi] \\ v(x,0) = \sin(x), & x \in [0, 2\pi] \end{cases} \tag{96}$$

我们选取初值 $u_0(x) = \sin x$，此时 $|u| \leqslant 1$. 我们在 $[0, 2\pi] \times [0, T]$ 上求解，选取 $\frac{\Delta t}{\Delta x} = 0.8$.

由前文分析得，间断在 $T = 1$ 开始形成。我们分别取 $T = 0.8$ 和 $T = 1.0$，使用若干种方法进行求解，结果如下：

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.15974285145033132	0.0000000000000000	0.17592741487409058	0.0000000000000000
32	5.2015122562181618E-002	1.6187483417156032	7.9461331340509256E-002	1.1466554458803440
64	1.2581050499842220E-002	2.0476787344371541	2.4376553897068193E-002	1.7047587781686333
128	2.6097663891206176E-003	2.2692598140880347	5.8678548532367647E-003	2.0545890970138863
256	4.7422010914903001E-004	2.4602919247899919	1.2482941012242899E-003	2.2328753082485400
512	9.1297010653480787E-005	2.3769173133321821	2.2452153009064335E-004	2.4750321754473612
1024	1.9237912711742131E-005	2.2466153456025730	4.2170883510245227E-005	2.4125346437193507
2048	4.4410123583139787E-006	2.1149917859711542	8.8170216363536724E-006	2.2578839423291774
4096	1.0690799294669375E-006	2.0545188662096785	1.9960926686546365E-006	2.1431127006022055
8192	2.6246673985654176E-007	2.0261632005991324	4.7326735094777028E-007	2.0764513952269215
16384	6.5038782133149458E-008	2.0127624661103969	1.1511813827191908E-007	2.0395402334008477

表 16: results for WENO3 with TVDRK2, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$, $T = 0.8$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.26570807071216773	0.0000000000000000	0.28213499908315220	0.0000000000000000
32	0.15498516517502242	0.77771191878663637	0.23229139965224299	0.28044990370383310
64	9.1685248175135736E-002	0.75736859745335339	0.19841593608460767	0.22740783817528681
128	5.4174130193497698E-002	0.75908554312917875	0.16789509326384761	0.24096783456422877
256	3.2082755771217983E-002	0.75580601651227619	0.14190434070535740	0.24264134750508831
512	1.9023638476299710E-002	0.75400486484025586	0.11959507912374823	0.24676069118526461
1024	1.1297929983690620E-002	0.75173473855121042	0.10064933747175805	0.24882035372710054
2048	6.7175600304299017E-003	0.75004925121620281	8.4571664459872020E-002	0.25109139790640406
4096	3.9968724226807310E-003	0.74906578502003718	7.0916510044807535E-002	0.25405283284125219
8192	2.3786926133672194E-003	0.74870267220294540	5.9315033140626666E-002	0.25772374385095642
16384	1.4155939688644255E-003	0.74876133120357047	4.9468961652362295E-002	0.26187417816558528

表 17: results for WENO3 with TVDRK2, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$, $T = 1.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.16719174202558823	0.0000000000000000	0.18698882547777157	0.0000000000000000
32	5.1707472993787279E-002	1.6930588858208457	8.1817092866280305E-002	1.1924778754387417
64	1.1442354992099002E-002	2.1759887913930815	2.3342352938517219E-002	1.8094522822928618
128	2.0242341141293789E-003	2.4989359482748510	4.8929447627231171E-003	2.2541750927004500
256	2.8796373280847522E-004	2.8134171255245857	8.7646523548035038E-004	2.4809342225221895
512	3.5901652708653118E-005	3.0037649606953112	1.1225350971363901E-004	2.9649363154852622
1024	4.3707191994441775E-006	3.0381076600326522	1.3788077267123944E-005	3.0252673592315107
2048	5.3618136582476202E-007	3.0270777073237305	1.6912821085568464E-006	3.0272320598019369
4096	6.6319578549512603E-008	3.0152143376245508	2.0884409401844550E-007	3.0176190729884405
8192	8.2442590830609300E-009	3.0079730901055957	2.5939302816802012E-008	3.0092147363455042
16384	1.0276586187695877E-009	3.0040287529458327	3.2317297726791594E-009	3.0047612295818102

表 18: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$, $T = 0.8$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.28031241593459938	0.0000000000000000	0.30637728143895870	0.0000000000000000
32	0.16047714164707869	0.80466783276904452	0.24828533819056164	0.30331025253051502
64	9.3434674509239368E-002	0.78033786213243650	0.20530091204979384	0.27425903299580023
128	5.4760071970925119E-002	0.77083370463021539	0.17102072375221150	0.26356887969537524
256	3.2262813560270158E-002	0.76325208487450680	0.14313261050296366	0.25681875216529798
512	1.9081675373417289E-002	0.75768441214591042	0.12009604559699871	0.25316375657636203
1024	1.1317030120877159E-002	0.75369243806752695	0.10085810025264350	0.25186169191359559
2048	6.7239357634658846E-003	0.75111756054569412	8.4660918417061928E-002	0.25255891181387979
4096	3.9989865437943381E-003	0.74967151535266641	7.0954887310437378E-002	0.25479408008176341
8192	2.3793933697436207E-003	0.74904062368838709	5.9331663973944959E-002	0.25809981510651686
16384	1.4158257216867810E-003	0.74895011232756703	4.9476173525635721E-002	0.26206831720116286

表 19: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$, $T = 1.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.12815240774464690	0.0000000000000000	0.14372581281147856	0.0000000000000000
32	3.5750166943639791E-002	1.8418387004160786	5.6843083046285603E-002	1.3382624791522320
64	6.4758712431684257E-003	2.4648037723211029	1.4578129741743884E-002	1.9631791601501958
128	6.5041480362793447E-004	3.3156423024166517	1.8094439089894054E-003	3.0101873538557942
256	3.9811070129697292E-005	4.0301185369400390	1.4073157714908358E-004	3.6845284064033437
512	1.8166609085771003E-006	4.4538085903667000	6.5618062752292872E-006	4.4227092628364995
1024	1.1755940051091584E-007	3.9498273441362950	3.5303804205022438E-007	4.2161974453039317
2048	1.2026105009536282E-008	3.2891485410216865	2.9240537630093044E-008	3.5937819085514695
4096	1.4350515223227547E-009	3.0669950213356034	3.2130330895885351E-009	3.1859620939262232

表 20: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$, $T = 0.8$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.21528892222142743	0.0000000000000000	0.24129314635919363	0.0000000000000000
32	0.12094964753324047	0.83186752187991864	0.19244219955716502	0.32636174392027967
64	7.1860041192165047E-002	0.75114489763248704	0.16210208114570812	0.24752258120461557
128	4.2283603849091717E-002	0.76509141887548637	0.13490587704266493	0.26494941374362074
256	2.5023216251818615E-002	0.75683111141424886	0.11289523713742866	0.25696857676759488
512	1.4908429386972184E-002	0.74713895530593644	9.5106487804637430E-002	0.24736895773612139
1024	8.9081766982084202E-003	0.74292619662648163	8.0345170645870823E-002	0.24333245004565984
2048	5.3218608623717420E-003	0.74319938274979802	6.7845335972304716E-002	0.24396166861162633
4096	3.1721944764011327E-003	0.74644957179329663	5.7138043262949814E-002	0.24779801008237134

表 21: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.8$, $T = 1.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.12152966989232257	0.0000000000000000	0.13596486851414147	0.0000000000000000
32	3.5322657128876292E-002	1.7826427939307270	5.6309627177645094E-002	1.2717804226668790
64	6.4688163310694978E-003	2.4490202170233677	1.4592403939062359E-002	1.9481640278574193
128	6.3557479862668168E-004	3.3473679247465280	1.7824303240394218E-003	3.0332999837543513
256	3.6417608291686946E-005	4.1253538388131279	1.3078940133071093E-004	3.7685281419713008
512	1.2815011803347387E-006	4.8287294646004710	5.0183707788198362E-006	4.7038827576980724
1024	3.8818432210603595E-008	5.0449491451963349	1.5827559526893076E-007	4.9867083405911128
2048	1.1607790957187668E-009	5.0635765034074423	4.7874687408855365E-009	5.0470319440806586
4096	3.5242537764762142E-011	5.0416318188601998	1.4409248794144958E-010	5.0541960374753625

表 22: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 0.8$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.21245733122424101	0.0000000000000000	0.23858756065865994	0.0000000000000000
32	0.12042810688943770	0.81900098323868953	0.19189123719155832	0.31422999493416903
64	7.1071868850530712E-002	0.76082160317527248	0.16031034358646118	0.25942131720620215
128	4.1971324476852527E-002	0.75987464513461400	0.13389680357361655	0.25974599384134917
256	2.4921123761706011E-002	0.75203486575075873	0.11242642743477049	0.25214031914302054
512	1.4871700635288559E-002	0.74479949043275284	9.4866716206994384E-002	0.24500728693002485
1024	8.8956181407038176E-003	0.74140287006138805	8.0228459268325003E-002	0.24178791784976589
2048	5.3179117336512940E-003	0.74223502765448524	6.7792936999726938E-002	0.24297911741623848
4096	3.1710592941754639E-003	0.74589497765377377	5.7116448792277799E-002	0.24722869137403855

表 23: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 1.0$

为了验证 WENO 格式的消除震荡的特性，我们将求解结果用 `matlab` 绘制成图像，并与由上文方法给出的真解的图像作对比[9]。不难发现，WENO 格式有效地消除了震荡。

以下是使用 TVDRK3 做时间离散，WENO3 做空间离散，CFL 数选取 0.8，分别在时刻 $T = 0.8, 1.0, 1.5, 5.0$，空间区间划分数量 $N = 32, 64$ 所求得的数值解，以及根据上述算法得到的真解的图像。

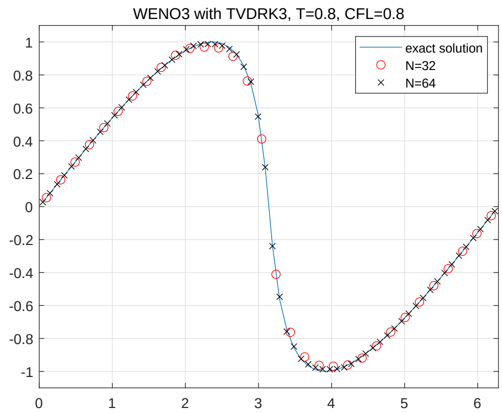

图 1: WENO3 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 0.8$

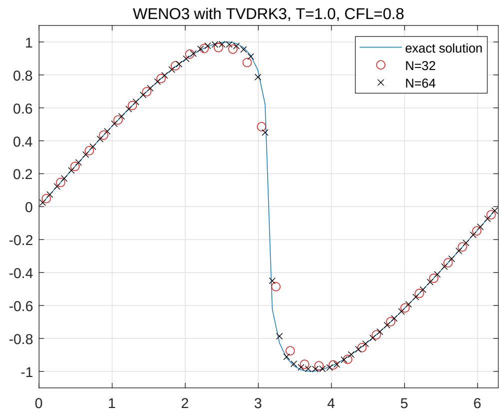

图 2: WENO3 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 1.0$

[9] 2021.6.24

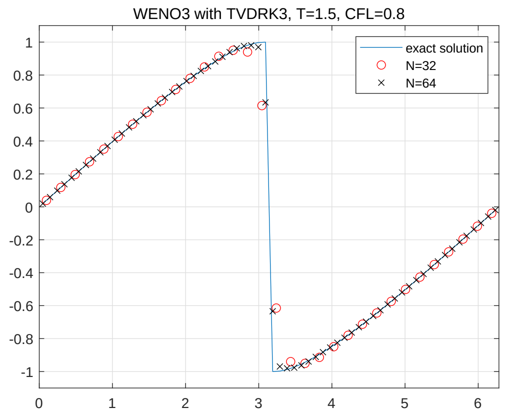

图 3: WENO3 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 1.5$

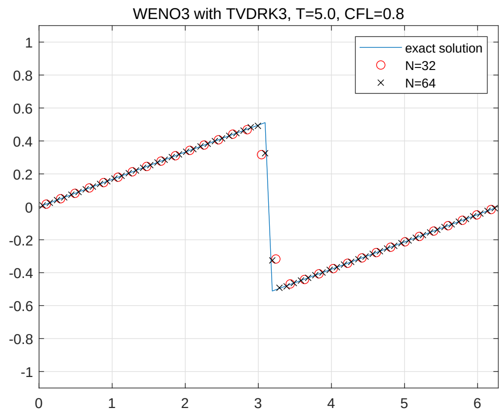

图 4: WENO3 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 5.0$

以下是使用 TVDRK3 做时间离散，WENO5 做空间离散，CFL 数选取 0.8，分别在时刻 $T = 0.8, 1.0, 1.5, 5.0$，空间区间划分数量 $N = 32, 64$ 所求得的数值解，以及根据上述算法得到的真解的图像。

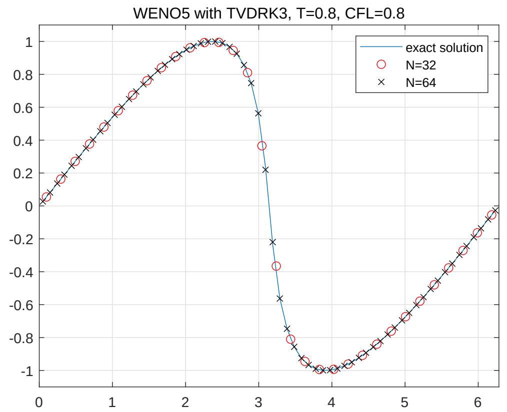

图 5: WENO5 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 0.8$

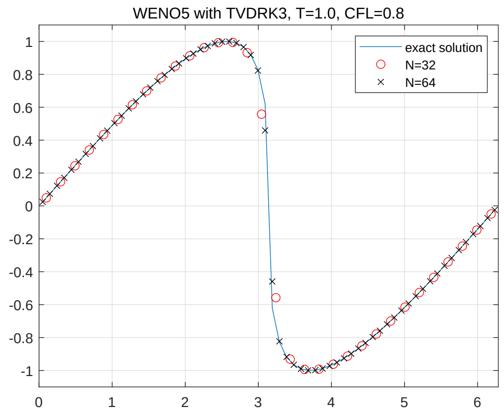

图 6: WENO5 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 1.0$

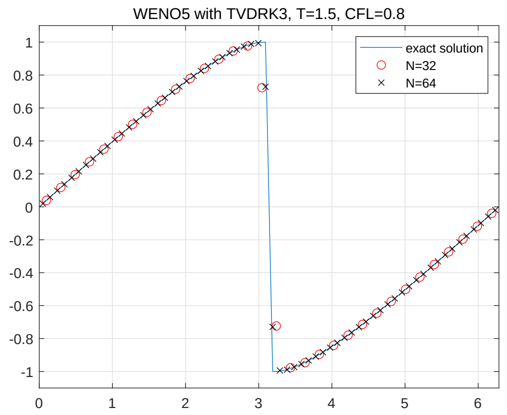

图 7: WENO5 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 1.5$

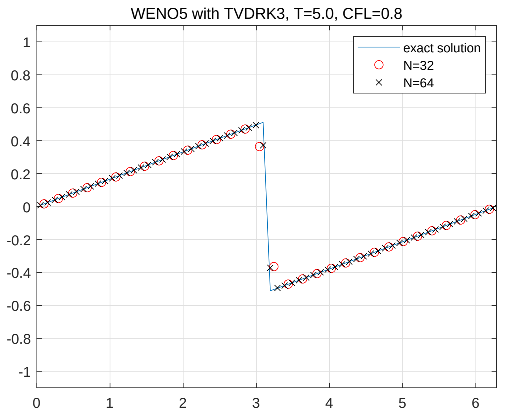

图 8: WENO5 with TVDRK3, $N = 32, 64$, $c = 0.8$, $T = 5.0$

7.4 Example 4–A Simple 2D Linear Case

我们先用一个线性方程验证数值方法的正确性[10]。考虑：

$$\begin{cases} u_t + u_x + u_y = 0, & t \geqslant 0, (x, y) \in [0, 2\pi]^2 \\ u_0(x, y) = u_0(x, y), & (x, y) \in [0, 2\pi]^2 \\ u(0, y, t) = u(2\pi, y, t),\ u(x, 0, t) = u(x, 2\pi, t), & t \geqslant 0 \end{cases} \tag{97}$$

易见该方程的解为 $u(x, y, t) = u_0(x - t, y - t)$，CFL 条件为 $\frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta y} \leqslant 1$. 我们取初值 $u_0(x, y) = \sin x \cos y$，终止时刻 $t = \pi$. 网格 $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$，此时 $c = 0.8$.

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.81441116635196920	0.0000000000000000	0.31610998654806954	0.0000000000000000
32	0.33214475516045727	1.2939452090186194	0.15826784970601726	0.99805839526289208
64	0.11575410362713050	1.5207487935242641	7.4094825335711212E-002	1.0949235226835969
128	3.1977031849459950E-002	1.8559554024511389	3.1261826138186954E-002	1.2449707356883368
256	3.6674357372155583E-003	3.1241924406110591	5.3283444128100488E-003	2.5526428115789801
512	3.5435885136474626E-004	3.3714886954334284	5.8354371107716130E-004	3.1907747061680158

表 24: results for WENO3 with TVDRK2, periodic boundary, $\epsilon = 10^{-6}$, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$

[10] 2021.6.24

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.71217694185689584	0.0000000000000000	0.31592458142252156	0.0000000000000000
32	0.24580907982894998	1.5346975185937224	0.14344234351306173	1.1391092307166453
64	7.5736238132649125E-002	1.6984825392159040	5.9680066015041278E-002	1.2651499272385671
128	2.0229922371765015E-002	1.9044929795608316	2.1448862174337058E-002	1.4763480141934104
256	3.4898789795744707E-003	2.5352418705595769	5.2689675273772663E-003	2.0253089237077031
512	2.7266325460696564E-004	3.6779848141431026	5.8172661792443847E-004	3.1791070645782979

表 25: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	3.3234679399866400E-002	0.0000000000000000	9.2640411967080949E-003	0.0000000000000000
32	2.7329927403828207E-003	3.6041359041623293	7.3331528498377097E-004	3.6591361530633857
64	2.9656680044568159E-004	3.2040526200711668	7.1191955237748239E-005	3.3646474812250431
128	3.5681752617535296E-005	3.0550987180129905	8.1787621690754264E-006	3.1217598071484427
256	4.4168388086629430E-006	3.0141003916562181	9.9851318957622226E-007	3.0340291223783336
512	5.5074895061499185E-007	3.0035474364118500	1.2408318639245408E-007	3.0084738450357627

表 26: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$

7.5 Example 5—2D Burgers Equation

考虑二维的无粘 Burgers 方程算例

$$\begin{cases} u_t + \frac{\partial}{\partial x}\frac{u^2}{2} + \frac{\partial}{\partial y}\frac{u^2}{2} = 0, & t \geqslant 0,\ (x,y) \in [0,2\pi]^2 \\ u(x,y,0) = \sin(x+y), & (x,y) \in [0,2\pi]^2 \\ u(0,y,t) = u(2\pi,y,t), u(x,0,t) = u(x,2\pi,t), & t \geqslant 0,\ 0 \leqslant x,y \leqslant 2\pi \end{cases} \tag{98}$$

作变量代换：$x = \xi + \eta, y = \xi - \eta$，令 $v(\xi,\eta,t) = u(x,y,t)$，那么 v 满足方程：

$$\begin{cases} v_t + vv_\xi = 0 \\ v(\xi,\eta,0) = \sin(2\xi) \end{cases} \tag{99}$$

易见，$v(\xi,\eta,t) = \widetilde{v}(\xi,t)$，这是因为 $\frac{\mathrm{d}}{\mathrm{d}\eta}v = 0$. 此时 $\widetilde{v}$ 满足方程：

$$\begin{cases} \widetilde{v}_t + \widetilde{v}\widetilde{v}_\xi = 0 \\ \widetilde{v}(\xi,0) = \sin(2\xi) \end{cases} \tag{100}$$

易见，$\widetilde{v}(\xi,t) = \varphi(2\xi,2t)$，其中 $\varphi(x,t)$ 是一维 Burgers 方程算例的解。

因此，我们给出上述算例的真解：$u(x,y,t) = \varphi(x+y,2t)$.[11] 由前文知，激波在 $t = 0.5$ 出现。

[11] 2021.6.25

由于 $\max|u| = 1$，所以我们令网格为：$\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$，此时 CFL 数为 0.8. 我们的结果如下表。[12]

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.35512678267822373	0.0000000000000000	0.13319033747308895	0.0000000000000000
32	0.10159287731203549	1.8055349103802141	3.8738209217418329E-002	1.7816602553975731
64	2.7705417599939421E-002	1.8745592402838851	1.2072512974959482E-002	1.6820312487991531
128	8.3494147101068911E-003	1.7304211387227186	4.6155676174813864E-003	1.3871460290912139
256	1.3665589134539582E-003	2.6111274119680017	1.0442594169602915E-003	2.1440279263003519
512	1.1993157538721376E-004	3.5102642127022454	8.9239407670804738E-005	3.5486554056385127
1024	1.2230113101014943E-005	3.2937018884094811	9.2998814510969474E-006	3.2623967067803310

表 27: results for WENO3 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 0.4$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.40373168262404646	0.0000000000000000	0.16831407631434947	0.0000000000000000
32	0.22454122549174893	0.84641645965588286	0.12805729375849173	0.39436640923562377
64	0.10115566831167173	1.1504031819679057	8.4918098751379123E-002	0.59264545092395149
128	6.6828448011525377E-002	0.59804288904180658	8.3074048811568879E-002	3.1674201494177628E-002
256	4.1208812474914806E-002	0.69750948049216710	7.2062208999832900E-002	0.20515499240338966
512	2.6281042323827247E-002	0.64893039464730506	6.4719355461059502E-002	0.15504563684949235
1024	1.6823189783645665E-002	0.64357121937817618	5.8186941126854019E-002	0.15350183410436058

表 28: results for WENO3 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 0.5$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.25788457946531057	0.0000000000000000	8.8720519966651534E-002	0.0000000000000000
32	0.13607049361231197	0.92237125030515210	7.7705812111285799E-002	0.19124530976956305
64	8.1778221520025465E-002	0.73456566446119087	7.3450611746939343E-002	8.1248004412281125E-002
128	6.2652785761929564E-002	0.38433802981271148	7.9768110669347303E-002	-0.11903760109146910
256	4.4707527728836245E-002	0.48686088941014549	8.0499134537271311E-002	-1.3161164925687144E-002
512	3.1778753336152370E-002	0.49245523920349638	8.0920734328919108E-002	-7.5361933363315753E-003
1024	2.2525874084236865E-002	0.49647944078884126	8.1118325453300666E-002	-3.5184587832554999E-003

表 29: results for WENO3 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 1.0$

[12] 2021.6.25

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	2.7418767571895868E-002	0.0000000000000000	9.5691269841821081E-003	0.0000000000000000
32	1.8035280123701303E-003	3.9262699894964141	5.7460820297439152E-004	4.0577368162610004
64	2.2645555606386393E-004	2.9935219899074226	7.5518898541404589E-005	2.9276689606363213
128	2.6651449203599877E-005	3.0869420477436180	9.6256997050847559E-006	2.9718744017843992
256	3.2351842229304959E-006	3.0422942114693305	1.2006291577604244E-006	3.0031008060233164
512	3.7355465582799149E-007	3.1144566184358666	1.3914458407704444E-007	3.1091339490569103
1024	4.8443715369476386E-008	2.9469379219598184	1.8086605102496378E-008	2.9435912144650227
2048	5.9464985509643570E-009	3.0261971879998533	2.2237324204255060E-009	3.0238665306663886

表 30: results for WENO5 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	2.7418767571895864E-002	0.0000000000000000	9.5691269841821081E-003	0.0000000000000000
32	1.8035280123701038E-003	3.9262699894964346	5.7460820297428050E-004	4.0577368162612801
64	5.7184337761406253E-005	4.9790579601431872	2.6719006418818303E-005	4.4266403231783285
128	2.5499204455264037E-006	4.4870959183128090	1.4354700729191094E-006	4.2182712020461768
256	8.0410499670083490E-008	4.9869245323492866	6.1790786487492255E-008	4.5379877064087282
512	1.6718713136524156E-009	5.5878481843630725	1.0174312503608007E-009	5.9243885190878407
1024	3.3044243102198129E-011	5.6609210447869263	1.3449685809518996E-011	6.2412150308947858

表 31: results for WENO5 with TVDRK3, periodic boundary, $\Delta x : \Delta y = 1 : 1$, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	8.1727716661836994E-002	0.0000000000000000	3.3407756340095074E-002	0.0000000000000000
32	2.5293274321738595E-002	1.6920716164585292	1.5591539267802879E-002	1.0994197302485789
64	1.0139927284663047E-002	1.3187065055492844	8.8213038066532468E-003	0.82169955457725319
128	1.5964539455229046E-003	2.6671044667610722	2.0000068418890016E-003	2.1409869694340879
256	9.9790442264198671E-005	3.9998254808051850	1.4688114083229176E-004	3.7672838608747710
512	4.5536981951756353E-006	4.4537910611787934	6.7883358375320224E-006	4.4354474184446193
1024	2.9467739555523215E-007	3.9498286912270548	3.5328532972667137E-007	4.2641521934488846

表 32: results for WENO5 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 0.4$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	9.4447571607005940E-002	0.0000000000000000	3.9219699332574187E-002	0.0000000000000000
32	6.6103385128811601E-002	0.51478954969161250	4.1840723155687320E-002	-9.3329311140019744E-002
64	3.2108556509662919E-002	1.0417663455689852	2.8872563480840774E-002	0.53520858222005652
128	2.0885126851814446E-002	0.62048190176147255	2.6554105432394670E-002	0.12076427641551486
256	1.4299714474137932E-002	0.54648956549989636	2.5698703966353265E-002	4.7239324590053200E-002
512	9.7934467271169924E-003	0.54609774108256259	2.4886409662950193E-002	4.6337494265036576E-002
1024	6.6248024709784688E-003	0.56393925549734758	2.3808303820921084E-002	6.3893267084102839E-002

表 33: results for WENO5 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 0.5$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	7.4380808939146531E-002	0.0000000000000000	2.9562404272792553E-002	0.0000000000000000
32	5.0129936853420626E-002	0.56925802145292548	3.0462548712070880E-002	-4.3273046125224410E-002
64	3.9121600899045539E-002	0.35770700864901650	3.5036007885972165E-002	-0.20180174611196830
128	2.9238170978085943E-002	0.42011234323311836	3.7226962286777909E-002	-8.7509498296274274E-002
256	2.0960715449613219E-002	0.48016510388239020	3.7742473518044273E-002	-1.9841078198088009E-002
512	1.4905753100796582E-002	0.49181869291508734	3.7957087082188923E-002	-8.1803051903962219E-003
1024	1.0568639843595330E-002	0.49607955056399672	3.8060365395257922E-002	-3.9201314149798520E-003

表 34: results for WENO5 with TVDRK3, periodic boundary, $\Delta x : \Delta y : \Delta t = 1 : 1 : 0.4$, $T = 1.0$

我们取 $T = 0.4, 0.5, 1.0$ 绘制数值解与真解的图像。

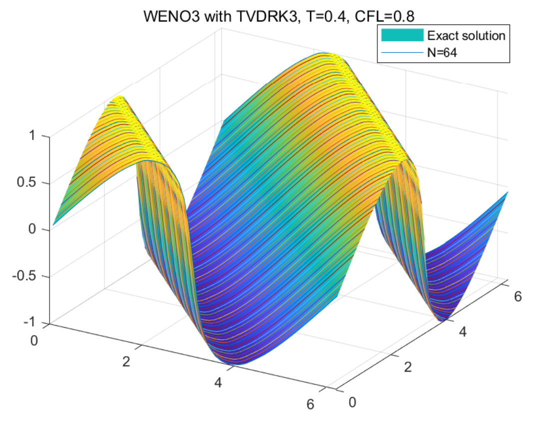

图 9: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.4$

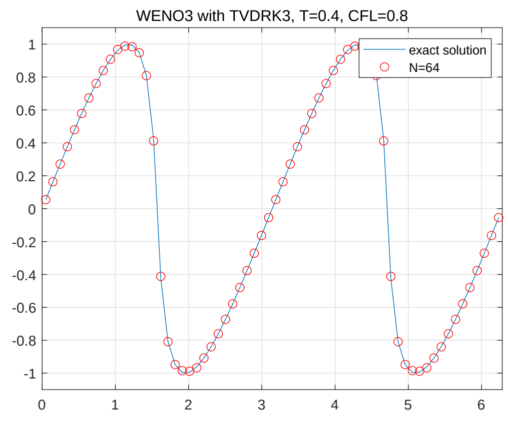

图 10: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.4$, 截面误差

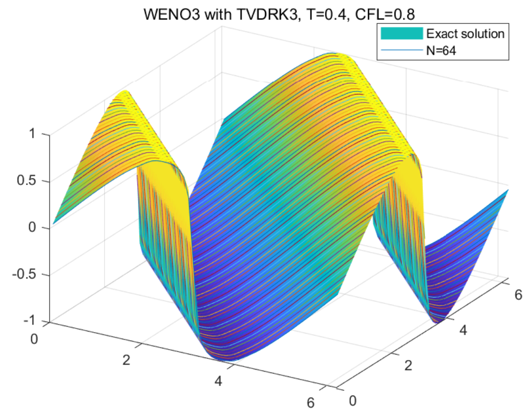

图 11: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.5$

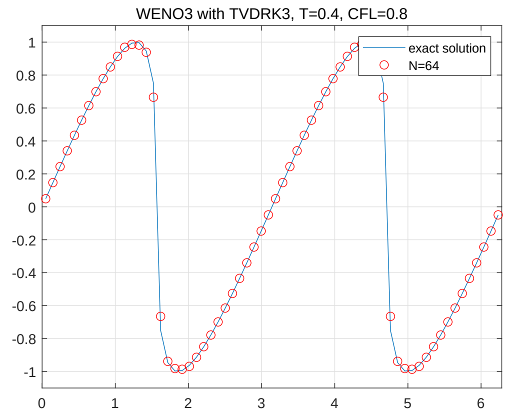

图 12: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.5$, 截面误差

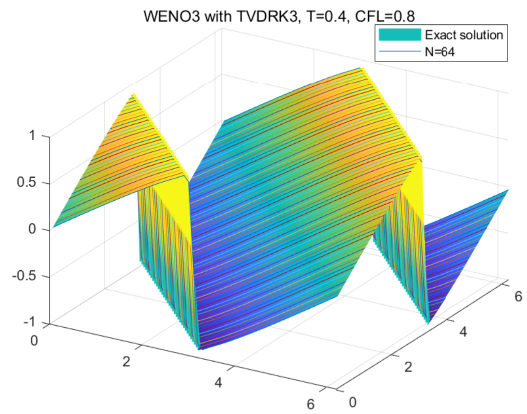

图 13: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 1.0$

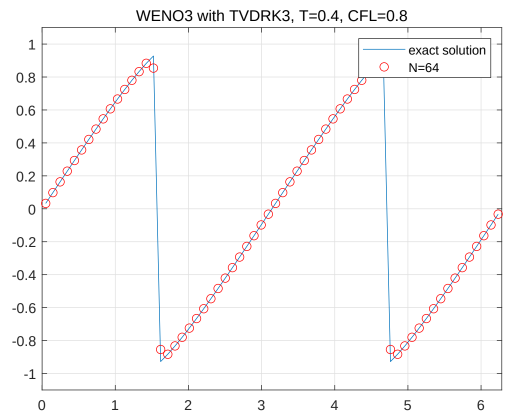

图 14: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 1.0$, 截面误差

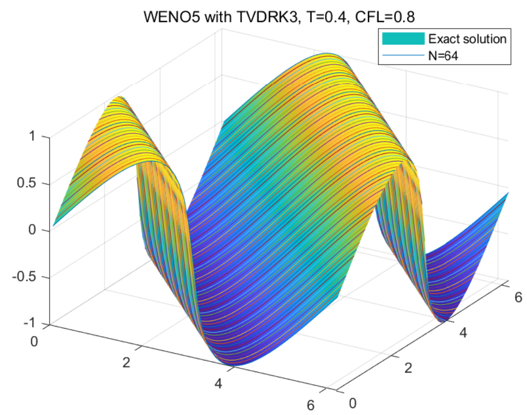

图 15: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.4$

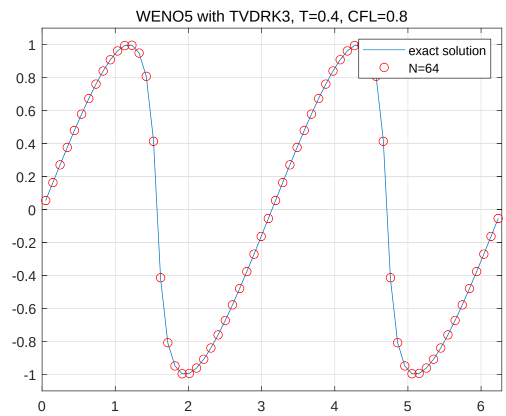

图 16: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.4$, 截面误差

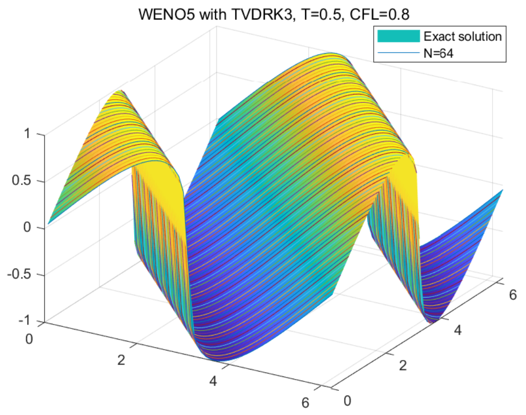

图 17: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.5$

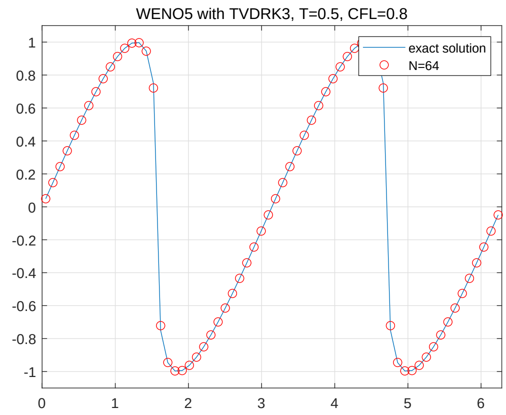

图 18: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 0.5$, 截面误差

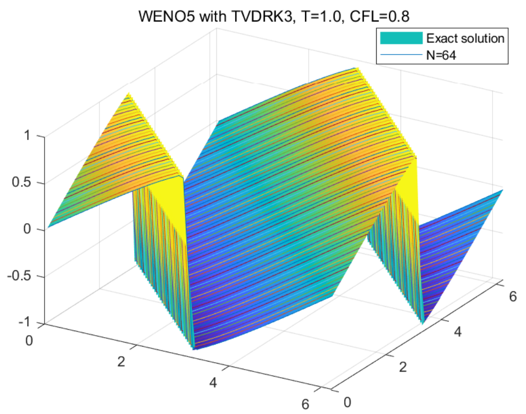

图 19: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 1.0$

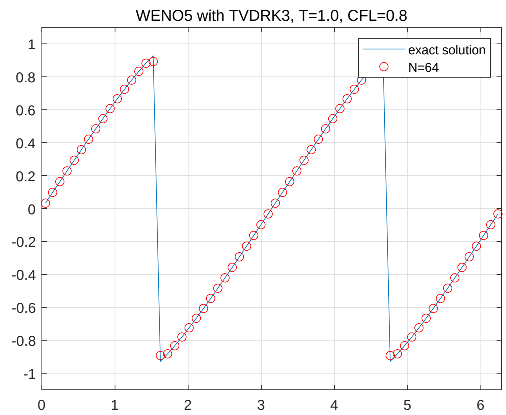

图 20: WENO3 with TVDRK3, $N = 32$, $c = 0.8$, $T = 1.0$, 截面误差

8 Examples for Euler System

8.1 Derivation for 1D Euler System

考虑 Euler 方程组 (守恒形式)：[13]

$$\xi = (\rho, \rho u, E)^T, \quad f(\xi) = \left(\rho u, \rho u^2 + p, u(E+p)\right)^T \tag{101}$$

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2, \quad \text{with } \gamma = 1.4 \text{ for air} \tag{102}$$

$$\xi_t + f(\xi)_x = 0 \tag{103}$$

[13] 2021.6.28

其中，声速 $c=\sqrt{\frac{\gamma p}{\rho}}$，我们记 $\xi=(\rho,\rho u,E)^T=(\xi_1,\xi_2,\xi_3)^T$，有换算关系：

$$\begin{cases}\rho=\xi_1\\ u=\frac{\xi_2}{\xi_1}\\ p=(\gamma-1)\left(\xi_3-\frac{\xi_2^2}{2\xi_1}\right)\end{cases}\qquad\begin{cases}\xi_1=\rho\\ \xi_2=\rho u\\ \xi_3=E=\frac{p}{\gamma-1}+\frac{\rho u^2}{2}\end{cases}\tag{104}$$

Jacobi 矩阵：

$$\begin{aligned}f'(\xi)=\frac{\partial\left(f_1,f_2,f_3\right)}{\partial\left(\xi_1,\xi_2,\xi_3\right)}&=\begin{pmatrix}0&1&0\\ \frac{(\gamma-3)\xi_2^2}{2\xi_1^2}&\frac{(3-\gamma)\xi_2}{\xi_1}&\gamma-1\\ \frac{(\gamma-1)\xi_2^3}{\xi_1^3}-\frac{\gamma\xi_2\xi_3}{\xi_1^2}&\frac{\gamma\xi_3}{\xi_1}-\frac{3(\gamma-1)\xi_2^2}{2\xi_1^2}&\frac{\gamma\xi_2}{\xi_1}\end{pmatrix}\\ &=\begin{pmatrix}0&1&0\\ \frac{(\gamma-3)u^2}{2}&(3-\gamma)u&\gamma-1\\ \frac{\gamma-2}{2}u^3-\frac{c^2u}{\gamma-1}&\frac{c^2}{\gamma-1}+\frac{3-2\gamma}{2}u^2&\gamma u\end{pmatrix}\end{aligned}\tag{105}$$

因为 $f'(\xi)$ 的特征多项式为：

$$\begin{aligned}\left|\lambda-f'(\xi)\right|=&\begin{vmatrix}\lambda&-1&0\\ -\frac{(\gamma-3)u^2}{2}&\lambda-(3-\gamma)u&-\gamma+1\\ -\frac{\gamma-2}{2}u^3+\frac{c^2u}{\gamma-1}&-\frac{c^2}{\gamma-1}-\frac{3-2\gamma}{2}u^2&\lambda-\gamma u\end{vmatrix}\\ =&(\lambda-u)\left(\lambda-(u+c)\right)\left(\lambda-(u-c)\right)\end{aligned}\tag{106}$$

所以在 $p\neq 0$ 时，可以显式地实对角化 $\boldsymbol{R}^{-1}(\xi)f'(\xi)\boldsymbol{R}(\xi)=\boldsymbol{\Lambda}(\xi)$：

$$\boldsymbol{\Lambda}(\xi)=\operatorname{diag}\left(u+c,u,u-c\right)\tag{107}$$

$$\boldsymbol{R}(\xi)=\begin{pmatrix}1&1&1\\ u+c&u&u-c\\ \frac{c^2}{\gamma-1}+\frac{1}{2}u^2+uc&\frac{1}{2}u^2&\frac{c^2}{\gamma-1}+\frac{1}{2}u^2-uc\end{pmatrix},\quad \det\boldsymbol{R}(\xi)=\frac{2\gamma pc}{\rho(1-\gamma)}\tag{108}$$

$$\boldsymbol{R}^{-1}(\xi)=\frac{1}{4\gamma pc}\begin{pmatrix}u(\rho uc(\gamma-1)-2\gamma p)&2(\gamma p+\rho uc(1-\gamma))&2c\rho(\gamma-1)\\ 2c(2\gamma p+\rho u^2(1-\gamma))&4\rho uc(\gamma-1)&4c\rho(1-\gamma)\\ u(2\gamma p+\rho uc(\gamma-1))&2(\rho uc(1-\gamma)-\gamma p)&2c\rho(\gamma-1)\end{pmatrix}\tag{109}$$

8.2 Derivation for 2D Euler System

二维无黏 Euler 方程为[14]：

$$\xi_t+f(\xi)_x+g(\xi)_y=0\tag{110}$$

$$\xi=(\rho,\rho u,\rho v,E)^T\tag{111}$$

$$f(\xi)=\left(\rho u,\rho u^2+p,\rho uv,u(E+p)\right)^T\tag{112}$$

$$g(\xi)=\left(\rho v,\rho uv,\rho v^2+p,v(E+p)\right)^T\tag{113}$$

[14] 2021.6.29

其中 ρ 为密度，(u,v) 为流速向量，$E=\frac{p}{\gamma-1}+\frac{\rho(u^2+v^2)}{2}$ 为能量密度，p 为压强，$c=\sqrt{\frac{\gamma p}{\rho}}$ 为音速，对于空气来说，$\gamma=1.4$.

我们记 $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)^T$，有换算关系：

$$U=\begin{pmatrix}\rho\\u\\v\\p\end{pmatrix}=\begin{pmatrix}\xi_1\\\frac{\xi_2}{\xi_1}\\\frac{\xi_3}{\xi_1}\\(\gamma-1)\left(\xi_4-\frac{\xi_2^2+\xi_3^2}{2\xi_1}\right)\end{pmatrix},\quad \xi=\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\\\xi_4\end{pmatrix}=\begin{pmatrix}\rho\\\rho u\\\rho v\\\frac{p}{\gamma-1}+\frac{\rho(u^2+v^2)}{2}\end{pmatrix} \tag{114}$$

Jacobi 矩阵分别为：

$$f'(\xi)=\begin{pmatrix}0&1&0&0\\\frac{\gamma-3}{2}u^2+\frac{\gamma-1}{2}v^2&(3-\gamma)u&(1-\gamma)v&\gamma-1\\-uv&v&u&0\\\frac{\gamma-2}{2}(u^3+uv^2)-\frac{c^2u}{\gamma-1}&\frac{3-2\gamma}{2}u^2+\frac{1}{2}v^2+\frac{c^2}{\gamma-1}&(1-\gamma)uv&\gamma u\end{pmatrix} \tag{115}$$

$$g'(\xi)=\begin{pmatrix}0&0&1&0\\-uv&v&u&0\\\frac{\gamma-1}{2}u^2+\frac{\gamma-3}{2}v^2&(1-\gamma)u&(3-\gamma)v&\gamma-1\\\frac{\gamma-2}{2}(v^3+u^2v)-\frac{c^2v}{\gamma-1}&(1-\gamma)uv&\frac{3-2\gamma}{2}v^2+\frac{1}{2}u^2+\frac{c^2}{\gamma-1}&\gamma v\end{pmatrix} \tag{116}$$

特征分解 $f'=\boldsymbol{R}_1\boldsymbol{\Lambda}_1\boldsymbol{R}_1^{-1},g'=\boldsymbol{R}_2\boldsymbol{\Lambda}_2\boldsymbol{R}_2^{-1}$ (利用 `matlab` 符号计算验证)：

$$\boldsymbol{\Lambda}_1=\operatorname{diag}(u,u,u+c,u-c) \tag{117}$$

$$\boldsymbol{\Lambda}_2=\operatorname{diag}(v,v,v+c,v-c) \tag{118}$$

$$\boldsymbol{R}_1=\begin{pmatrix}1&0&\frac{\rho}{\sqrt{2}c}&\frac{\rho}{\sqrt{2}c}\\u&0&\frac{\rho}{\sqrt{2}c}(u+c)&\frac{\rho}{\sqrt{2}c}(u-c)\\v&-\rho&\frac{\rho v}{\sqrt{2}c}&\frac{\rho v}{\sqrt{2}c}\\\frac{u^2+v^2}{2}&-\rho v&\frac{\rho}{2\sqrt{2}c}(u^2+v^2)+\frac{\rho c}{\sqrt{2}(\gamma-1)}+\frac{\rho u}{\sqrt{2}}&\frac{\rho}{2\sqrt{2}c}(u^2+v^2)+\frac{\rho c}{\sqrt{2}(\gamma-1)}-\frac{\rho u}{\sqrt{2}}\end{pmatrix} \tag{119}$$

$$\boldsymbol{R}_2=\begin{pmatrix}1&0&\frac{\rho}{\sqrt{2}c}&\frac{\rho}{\sqrt{2}c}\\u&-\rho&\frac{\rho u}{\sqrt{2}c}&\frac{\rho u}{\sqrt{2}c}\\v&0&\frac{\rho}{\sqrt{2}c}(v+c)&\frac{\rho}{\sqrt{2}c}(v-c)\\\frac{u^2+v^2}{2}&-\rho u&\frac{\rho}{2\sqrt{2}c}(u^2+v^2)+\frac{\rho c}{\sqrt{2}(\gamma-1)}+\frac{\rho v}{\sqrt{2}}&\frac{\rho}{2\sqrt{2}c}(u^2+v^2)+\frac{\rho c}{\sqrt{2}(\gamma-1)}-\frac{\rho v}{\sqrt{2}}\end{pmatrix} \tag{120}$$

$$\boldsymbol{R}_1^{-1}=\begin{pmatrix}1-(\gamma-1)\frac{u^2+v^2}{2c^2}&\frac{(\gamma-1)u}{c^2}&\frac{(\gamma-1)v}{c^2}&\frac{1-\gamma}{c^2}\\\frac{v}{\rho}&0&-\frac{1}{\rho}&0\\(\gamma-1)\frac{u^2+v^2}{2\sqrt{2}\rho c}-\frac{u}{\sqrt{2}\rho}&\frac{1}{\sqrt{2}\rho}-\frac{(\gamma-1)u}{\sqrt{2}\rho c}&-\frac{(\gamma-1)v}{\sqrt{2}\rho c}&\frac{\gamma-1}{\sqrt{2}\rho c}\\(\gamma-1)\frac{u^2+v^2}{2\sqrt{2}\rho c}+\frac{u}{\sqrt{2}\rho}&-\frac{1}{\sqrt{2}\rho}-\frac{(\gamma-1)u}{\sqrt{2}\rho c}&-\frac{(\gamma-1)v}{\sqrt{2}\rho c}&\frac{\gamma-1}{\sqrt{2}\rho c}\end{pmatrix} \tag{121}$$

$$\boldsymbol{R}_2^{-1}=\begin{pmatrix}1-(\gamma-1)\frac{u^2+v^2}{2c^2}&\frac{(\gamma-1)u}{c^2}&\frac{(\gamma-1)v}{c^2}&\frac{1-\gamma}{c^2}\\\frac{u}{\rho}&-\frac{1}{\rho}&0&0\\(\gamma-1)\frac{u^2+v^2}{2\sqrt{2}\rho c}-\frac{v}{\sqrt{2}\rho}&-\frac{(\gamma-1)u}{\sqrt{2}\rho c}&\frac{1}{\sqrt{2}\rho}-\frac{(\gamma-1)v}{\sqrt{2}\rho c}&\frac{\gamma-1}{\sqrt{2}\rho c}\\(\gamma-1)\frac{u^2+v^2}{2\sqrt{2}\rho c}+\frac{v}{\sqrt{2}\rho}&-\frac{(\gamma-1)u}{\sqrt{2}\rho c}&-\frac{1}{\sqrt{2}\rho}-\frac{(\gamma-1)v}{\sqrt{2}\rho c}&\frac{\gamma-1}{\sqrt{2}\rho c}\end{pmatrix} \tag{122}$$

8.3 Example 1—1D Euler System—A Simple Case

在初始条件

$$\rho(x,0) = 1 + 0.2\sin(\pi x) \tag{123}$$

$$u(x,0) = 0.7 \tag{124}$$

$$p(x,0) = 1 \tag{125}$$

以及以 2 为周期的周期边界条件下，精确解为：

$$\rho(x,0) = 1 + 0.2\sin\left(\pi(x-0.7t)\right) \tag{126}$$

$$u(x,t) = 0.7 \tag{127}$$

$$p(x,t) = 1 \tag{128}$$

我们使用改进的 Component-wise WENO 进行数值计算。由于 f' 的谱半径

$$\sigma(\boldsymbol{R}(\xi)) = u + c = u + \sqrt{\frac{\gamma p}{\rho}} \leqslant 0.7 + \sqrt{\frac{0.56}{0.8}} \approx 1.537$$

所以选取 $\frac{\Delta t}{\Delta x} = 0.5$，则 CFL 数 $c \approx 0.768$. 选取终止时刻 $T = 2.0$.[15]

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	5.0660345118482497E-002	0.0000000000000000	6.1871147997490272E-002	0.0000000000000000
32	1.9574886204968774E-002	1.3718529846019230	2.4913276440476917E-002	1.3123520344904511
64	5.1675692420877122E-003	1.9214462006222512	8.1723251458374335E-003	1.6080962587185608
128	8.0592887425400645E-004	2.6807613878309313	1.7175218720655305E-003	2.2504181331813644
256	5.5713103200938579E-005	3.8545639410484434	1.4660767989749246E-004	3.5502958866896308
512	2.5800537497301737E-006	4.4325436494127359	5.5074437887991934E-006	4.7344340040990520
1024	1.1478304773880239E-007	4.4904196295106846	1.9397921824371167E-007	4.8274088610595935
2048	5.9027159882423924E-009	4.2813868489291886	8.7169873541625975E-009	4.4759286722415297
4096	4.9643872852137539E-010	3.5716913540590771	6.0501670340329383E-010	3.8487827384146915
8192	5.6934086732338834E-011	3.1242511030821389	6.1519234151319324E-011	3.2978655245916944

表 35: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.5$

[15] 2021.6.28

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	1.5578616867487536E-003	0.0000000000000000	1.6068514874638673E-003	0.0000000000000000
32	6.2476324797766214E-005	4.6401137518773057	6.9677972123871612E-005	4.5273901474949616
64	3.0261346854828969E-006	4.3677614839440979	3.2743306346283418E-006	4.4114307225865890
128	2.3228035925735695E-007	3.7035371240169250	2.3415085559364002E-007	3.8056897944993300
256	2.4569299926046264E-008	3.2409385150432226	2.4104608753461321E-008	3.2800573913175279
512	2.9388187633079282E-009	3.0635484543790552	2.9108402355149110E-009	3.0498014503308650
1024	3.6377659465865760E-010	3.0141117625909444	3.6337666209362851E-010	3.0018979874089360
2048	4.5424742962963312E-011	3.0015024670845585	4.5669468207165664E-011	2.9921638696672015
4096	5.8045648304706071E-012	2.9682185338704490	6.2436722458869554E-012	2.8707632786546688
8192	1.9796914630936108E-012	1.5519123096230953	1.8496315590255108E-012	1.7551568878501609

表 36: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.5$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	1.4972494808437786E-003	0.0000000000000000	1.5761556864133386E-003	0.0000000000000000
32	5.2233990649708546E-005	4.8411818927797938	5.9977602469896141E-005	4.7158423818887112
64	1.5965662275567531E-006	5.0319446250212270	1.8754576491808450E-006	4.9991092632890597
128	4.8125429742644471E-008	5.0520291652159317	5.4906128199583293E-008	5.0941316940537735
256	1.4081616161131108E-009	5.0949145953570749	1.4946457405073943E-009	5.1990916968719629
512	3.8120202941944543E-011	5.2071133130036058	4.0837999648601908E-011	5.1937475675353220
1024	7.3839986874154819E-012	2.3680815981557179	6.1419758168312910E-012	2.7331374639642552
2048	2.3512030213435834E-011	-1.6709249163535536	1.8153922809460710E-011	-1.5635065911515977

表 37: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta t = \Delta x^{\frac{5}{3}}$

设置的条件均与上文相同，我们使用 Characteristic-wise 方法进行计算。[16]

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	3.4949753987837472E-002	0.0000000000000000	4.5611830095545258E-002	0.0000000000000000
32	1.3906239874844231E-002	1.3295499199330270	1.8640165118796848E-002	1.2909934172247559
64	3.5265832819909660E-003	1.9793893653613672	5.8886167334790684E-003	1.6624139573265531
128	4.6325530720411396E-004	2.9283917006187052	1.0416437116589350E-003	2.4990668810179701
256	2.6623263227378512E-005	4.1210481852422527	6.6075145906507871E-005	3.9786103804693904
512	1.1978105745433867E-006	4.4742157349411622	2.3011020597873966E-006	4.8437108282910879
1024	5.3307991460941397E-008	4.4899041384659100	8.5114673753494685E-008	4.7567732882840028
2048	3.0698497393380233E-009	4.1181118787324236	4.2976957548290784E-009	4.3077726143809061
4096	2.9324020457756868E-010	3.3880132191327936	3.3955416256503668E-010	3.6618497307928761

表 38: results for WENO3 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.5$

[16] 2021.6.28

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	1.0653843818268693E-003	0.0000000000000000	1.1485927940686480E-003	0.0000000000000000
32	4.2961309552663393E-005	4.6321922560960527	4.8265778356926248E-005	4.5727229609135200
64	2.4231328507688917E-006	4.1480925611056625	2.6196101821085449E-006	4.2035765958168465
128	2.1438006962915967E-007	3.4986308107608379	2.1586721388189289E-007	3.6011360965408792
256	2.4058527208992739E-008	3.1555505554757532	2.3578761831899442E-008	3.1945842754937086
512	2.9247388507947543E-009	3.0401686100683278	2.9083728758649841E-009	3.0192038106202532
1024	3.6344039969316993E-010	3.0085171103419910	3.6329717012506535E-010	3.0009902126126575
2048	4.5418516645739365E-011	3.0003663031432328	4.5667691850326264E-011	2.9919043484747858
4096	5.8038816281647714E-012	2.9681905882175261	6.2414517998377050E-012	2.8712203215296661

表 39: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\frac{\Delta t}{\Delta x} = 0.5$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	1.0086059842819147E-003	0.0000000000000000	1.0840991496685604E-003	0.0000000000000000
32	3.2784133521311984E-005	4.9432211148378835	3.8576107362686685E-005	4.8126453273945824
64	9.8629250774625108E-007	5.0548383804874684	1.2203149775125155E-006	4.9823820939161223
128	2.9850412974776936E-008	5.0461927783336202	3.3817302069749644E-008	5.1733481952957190
256	8.8161925357376112E-010	5.0814513497223794	9.3582119831125965E-010	5.1753848473707649
512	2.3267288731004910E-011	5.2437807184683400	2.5209612175558505E-011	5.2141870800617820
1024	7.3245711853302483E-012	1.6674869025089807	5.8919535916857058E-012	2.0971559518999876

表 40: results for WENO5 with TVDRK3, periodic boundary, $\epsilon = 10^{-6}$, $\Delta t = \Delta x^{\frac{5}{3}}$

由实验结果可见，解的收敛阶与一维标量情形相类似。同时，Characteristic-wise 的计算开销明显大于 Component-wise 的计算开销。

8.4 Example 2—1D Euler System—Shock Tube Problem

考虑一维激波管问题[17]，在 $\mathbb{R}$ 上初值为：

$$(\rho, u, p) = \begin{cases} (0.445, 0.698, 3.528), & x < 0 \\ (0.5, 0, 0.571), & x > 0 \end{cases} \tag{129}$$

对于给定终止时刻 $T = 1.3$，由于波速有限，我们在 $[-5, 5]$ 上求解，在 $-5, 5$ 之外的点上均始终取初值。选取 $\frac{\Delta t}{\Delta x} = 0.2$，估算得 CFL 数 $c \leqslant 0.806$.

我们使用 WENO3 做空间离散，TVDRK3 做时间离散，方法采用 Characteristic-wise，使用 Lax-Friedrichs 通量分裂，计算获得以下结果。

[17] 2021.6.29

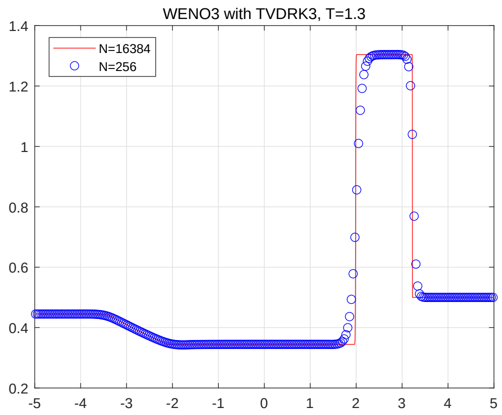

图 21: Density, $T = 1.3$

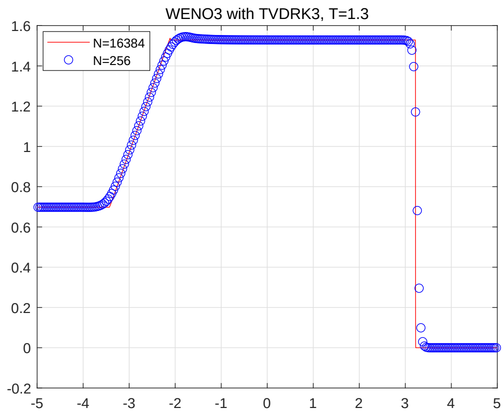

图 22: Velocity, $T = 1.3$

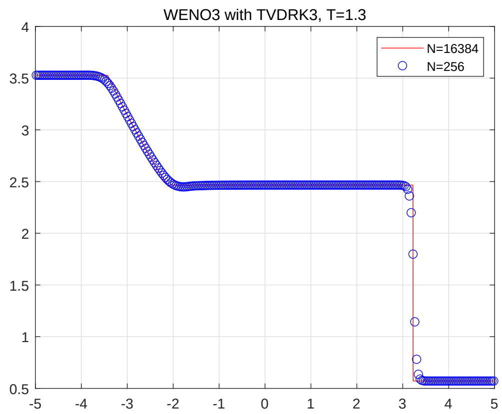

图 23: Pressure, $T = 1.3$

8.5 Example 3—2D Euler Syetem—Vortex Evolution

考虑 $[0,10]^2$ 上的二维 Euler 系统，在均匀流 $(u,v)=1, p=\rho=1$ 上，我们施加一个等熵涡流[18]：

$$(\delta u, \delta v) = \frac{\epsilon}{2\pi} \mathrm{e}^{\frac{1-r^2}{2}} (-\bar{y}, \bar{x}) \tag{130}$$

$$\delta T = \delta \frac{p}{\rho} = -\frac{(\gamma-1)\epsilon^2}{8\gamma\pi^2} \mathrm{e}^{1-r^2} \tag{131}$$

$$\delta S = \delta \frac{p}{\rho^\gamma} = 0 \tag{132}$$

其中 $(\bar{x}, \bar{y}) = (x-5, y-5)$, $r = \sqrt{\bar{x}^2 + \bar{y}^2}$，涡流强度 $\epsilon = 5$. 取周期边界条件。

易见，均匀流流过一个周期的时间为 $T = 10$. 由于有估计：$\sigma(f') \leqslant 2 + \frac{5}{2\pi} = 2.796$, $\sigma(g') \leqslant 2.796$，所以根据 CFL 条件，取 $\frac{\Delta t}{\Delta x} = 0.12$，那么 $c \leqslant 0.671$.

[18] 2021.6.30

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	9.6839462557988518E-002	0.0000000000000000	8.7406484260382888E-002	0.0000000000000000
32	3.7884920882064771E-002	1.3539713376494680	3.8815009502376041E-002	1.1711256694521881
64	1.3147454405488665E-002	1.5268402413501605	1.9865106149724088E-002	0.96637813822017204
128	3.8788603202437036E-003	1.7610787639288299	6.8416702505611493E-003	1.5378160255122264
256	9.7761584009366091E-004	1.9882932584858934	1.5752465092767620E-003	2.1187709592359449
512	1.7020314529247346E-004	2.5220099632633164	3.6875566463301368E-004	2.0948404960885147

表 41: results for WENO3 with TVDRK3, density, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	7.5069808607432301E-002	0.0000000000000000	8.5544923834087516E-002	0.0000000000000000
32	2.3029009380169139E-002	1.7047804515630693	2.6144590440927096E-002	1.7101697785894154
64	6.8577279003709628E-003	1.7476497857061717	1.0551973473627685E-002	1.3089996273721594
128	1.9436679217450100E-003	1.8189489088015456	3.7679279791085207E-003	1.4856695485678728
256	5.0280512937015244E-004	1.9507104810169780	1.0906292609882406E-003	1.7886106214568749
512	8.5681658082857923E-005	2.5529410634733596	1.8369001144913355E-004	2.5698156838508468

表 42: results for WENO3 with TVDRK3, pressure, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.56975650508442199	0.0000000000000000	0.37360733408748825	0.0000000000000000
32	0.40837761703585967	0.48044169625541877	0.30855382223689121	0.27600060990394365
64	0.15217593232264859	1.4241635903468426	0.10213816989357893	1.5950000519432646
128	3.0741871398138908E-002	2.3074633084214211	2.4608156440213858E-002	2.0533136291318934
256	5.5651626072267153E-003	2.4657092440635626	6.4182854618750063E-003	1.9388767180543320
512	7.4986838553269192E-004	2.8917145361776084	1.2284616910303026E-003	2.3853350881773778

表 43: results for WENO3 with TVDRK3, density, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 10.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.22329903080112620	0.0000000000000000	0.18467626201425136	0.0000000000000000
32	0.21026282449885528	8.6783192313661453E-002	0.19683605464630183	-9.1996068129495581E-002
64	9.8598961345882849E-002	1.0925494427788991	9.4171347775413139E-002	1.0636344220199567
128	2.4181606136890847E-002	2.0276623778958736	1.9968579935044084E-002	2.2375564383492770
256	5.4520822003984578E-003	2.1490308524903612	6.7449467044284894E-003	1.5658527904196577
512	8.4595275276804752E-004	2.6881583187740974	1.2199914184101157E-003	2.4669360439778671

表 44: results for WENO3 with TVDRK3, pressure, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 10.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	2.9327074469659598E-002	0.0000000000000000	2.1286873573712284E-002	0.0000000000000000
32	4.4148547657897495E-003	2.7317952779916750	5.3522936521597231E-003	1.9917348988172445
64	4.1994981180243682E-004	3.3940771518927337	5.3992307549333152E-004	3.3093314894130530
128	1.3363139273298232E-005	4.9738860505662021	2.5822808631503769E-005	4.3860360469180106
256	3.0234718506528227E-007	5.4659090086522841	4.0132990819508052E-007	6.0077134407686277
512	1.2019251109016646E-008	4.6527872341064889	9.8047749919061289E-009	5.3551603408157415

表 45: results for WENO5 with TVDRK3, density, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	1.9292008398209097E-002	0.0000000000000000	1.7045868262931863E-002	0.0000000000000000
32	3.5119960202536060E-003	2.4576402281031600	4.5435198152036227E-003	1.9075398118346099
64	3.0088927205189760E-004	3.5449866353811852	3.8150116351887142E-004	3.5740510085780128
128	9.8249727729820185E-006	4.9366354499401499	1.5899927632734290E-005	4.5845953527361152
256	2.6026765650901059E-007	5.2383854652541144	4.2198689631511854E-007	5.2356781884303132
512	8.8926374580786016E-009	4.8712408588404195	1.0152834795107424E-008	5.3772436937636297

表 46: results for WENO5 with TVDRK3, pressure, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.41542187419334298	0.0000000000000000	0.29204438290691137	0.0000000000000000
32	6.5238229033875439E-002	2.6707876597348954	4.4540951391746186E-002	2.7129833572501894
64	5.1103212649644568E-003	3.6742317235252306	6.9736192635767580E-003	2.6751528692991573
128	1.8025098011099249E-004	4.8253349811167219	1.5245083898474121E-004	5.5154916048276599
256	6.8184399991760081E-006	4.7244215931845766	5.2967511569024239E-006	4.8470925478035198
512	2.7069445243044367E-007	4.6547044740990593	2.0382665766316421E-007	4.6996930787453275

表 47: results for WENO5 with TVDRK3, density, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 10.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	0.23017497661043451	0.0000000000000000	0.21732406149993799	0.0000000000000000
32	4.7743166870950308E-002	2.2693648292403696	4.5900768193108843E-002	2.2432577106099476
64	9.5483201525132389E-003	2.3219754200650939	2.1030859250579015E-002	1.1260105037491772
128	3.2144202610899920E-004	4.8926164724924011	3.1209959835420964E-004	6.0743574841115784
256	1.0410138455688517E-005	4.9484974009492229	1.0528715769453356E-005	4.8896051198421766
512	3.6758382234956733E-007	4.8237721726098286	3.8628859355682721E-007	4.7685065875762929

表 48: results for WENO5 with TVDRK3, pressure, $\frac{\Delta t}{\Delta x} = 0.12$, $T = 10.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	NaN	0.0000000000000000	NaN	0.0000000000000000
32	1.4390355440002429E-002	NaN	1.9244800176361765E-002	NaN
64	4.3590317466142721E-004	5.0449507058750420	5.0085975203018052E-004	5.2639181966271993
128	1.3598784010673556E-005	5.0024581527634071	2.6267400189738943E-005	4.2530613622858198
256	3.0079835992807211E-007	5.4985371418144391	3.9899243886587499E-007	6.0407682000686505
512	9.8571607829214519E-009	4.9314807323246104	9.1268870239247235E-009	5.4500947233404347

表 49: results for WENO5 with TVDRK3, density, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	NaN	0.0000000000000000	NaN	0.0000000000000000
32	6.0172201543673837E-003	NaN	6.2766722965437172E-003	NaN
64	2.9606585409491250E-004	4.3451071260436489	3.8648747830927199E-004	4.0215063095675418
128	9.8494905704526449E-006	4.9097251925189704	1.5929002091130684E-005	4.6006938796005734
256	2.5940450654290618E-007	5.2467736599243020	4.2109062237116746E-007	5.2413813312797659
512	7.3991272154864463E-009	5.1317046293194117	8.8398397490863090E-009	5.5739667207212751

表 50: results for WENO5 with TVDRK3, pressure, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 0.2$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	NaN	0.0000000000000000	NaN	0.0000000000000000
32	NaN	NaN	NaN	NaN
64	5.2925341701750605E-003	NaN	6.4248789992296662E-003	NaN
128	1.8876214586898441E-004	4.8093172962210060	1.5387350938511624E-004	5.3838524979491345
256	6.7561150391739793E-006	4.8042317734568822	5.2565502988155544E-006	4.8714847540488853
512	2.1857375867436478E-007	4.9500017829399949	1.6833015092920789E-007	4.9647507994077165

表 51: results for WENO5 with TVDRK3, density, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 10.0$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	NaN	0.0000000000000000	NaN	0.0000000000000000
32	NaN	NaN	NaN	NaN
64	9.2561466617298756E-003	NaN	1.9876930136921822E-002	NaN
128	3.2598557436920101E-004	4.8275316955090313	3.0963727373378713E-004	6.0043719972815586
256	1.0345679555954003E-005	4.9777078063637790	1.0475609578453060E-005	4.8854730494152694
512	3.0893127819101348E-007	5.0655986554155579	3.3029929014105619E-007	4.9871165178674284

表 52: results for WENO5 with TVDRK3, pressure, $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 10.0$

注记[19]：

[19] 2021.7.4

- 最后一个算例规模很大，耗时很久，使用 `Intel i7-9750H` 超频 3.35GHz，`OpenMP` 12 线程并行，计算约 3.7 小时才完成。
- 由于该算例中，周期边界条件实际上是不精确的，所以会存在一定的固有误差。

使用 WENO5 做空间离散，TVDRK3 做时间离散，$\frac{\Delta t}{\Delta x} = 0.12$，$N = 256$，求解并绘制 $T = 0.2$ 和 $T = 10.0$ 时密度和压强的图像[20]。

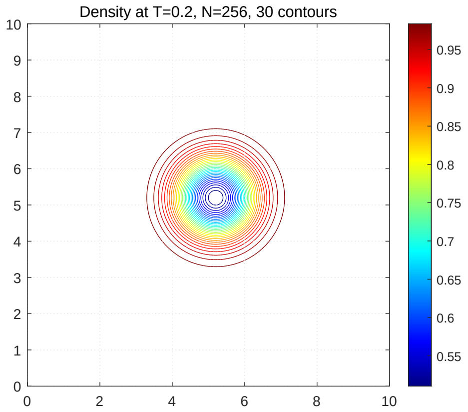

图 24: density, $T = 0.2$, $N = 256$

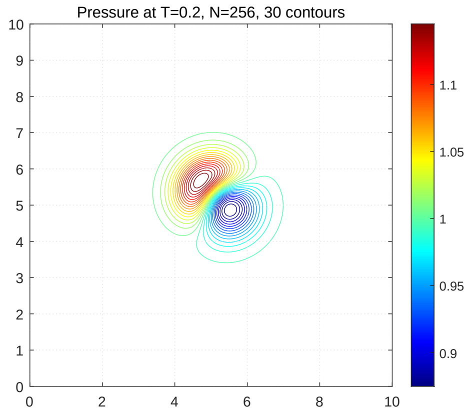

图 25: pressure, $T = 0.2$, $N = 256$

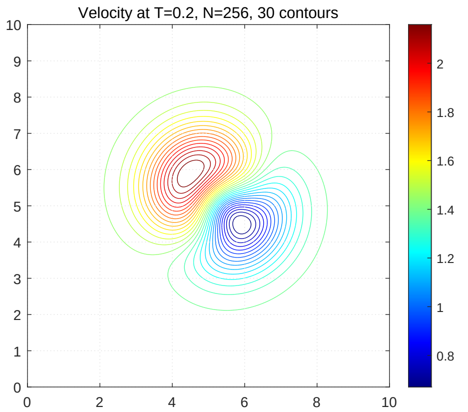

图 26: velocity, $T = 0.2$, $N = 256$

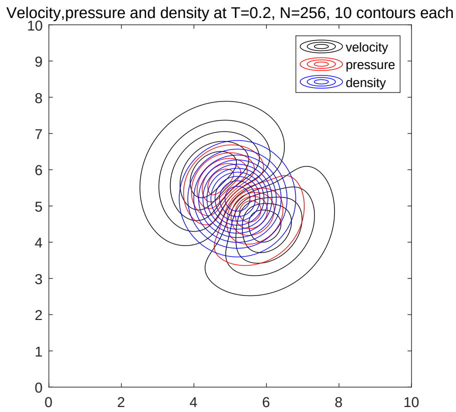

图 27: comparison, $T = 0.2$, $N = 256$

[20] 2021.7.1

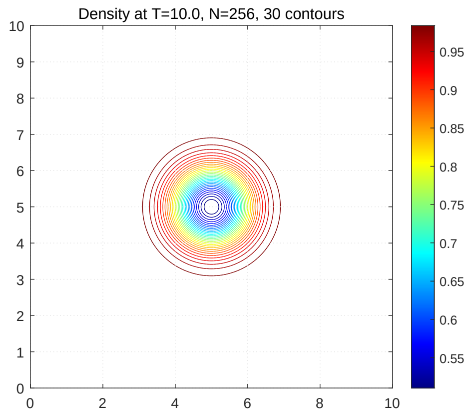

图 28: density, $T = 10.0$, $N = 256$

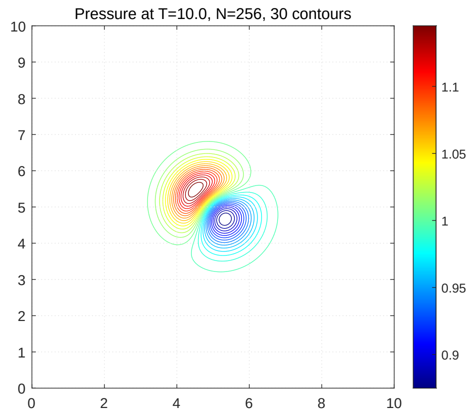

图 29: pressure, $T = 10.0$, $N = 256$

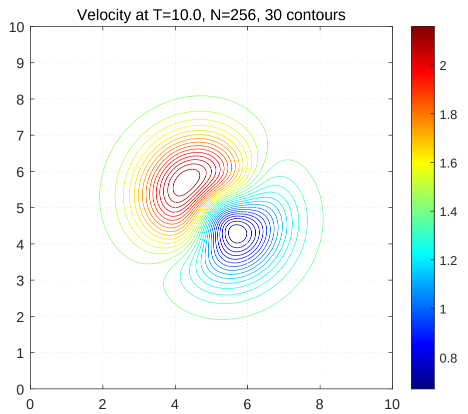

图 30: velocity, $T = 10.0$, $N = 256$

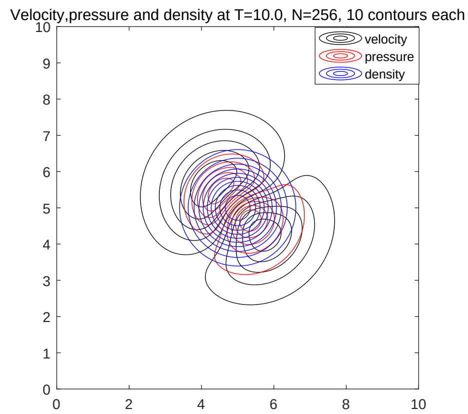

图 31: comparison, $T = 10.0$, $N = 256$

9 Examples for Hamilton Jacobi Equation

注记：下文数值实验中，若不加特殊说明，空间离散均采用标准 WENO5，时间离散均采用 TVDRK3。

9.1 Example 1—A Simple 1D Linear Case

为了验证程序的正确性，我们首先计算一个简单的方程[21]：

$$\begin{cases} \varphi_t + \varphi_x = 0, & x \in [-\pi, \pi] \\ \varphi(x, 0) = \sin x \\ \varphi(-\pi, t) = \varphi(\pi, t) \end{cases} \tag{133}$$

取终止时刻 $T = 20$. 取 Lax-Friedrichs 分裂为迎风 Hamiltonian。易见，数值实验结果符合预期。

[21] 2021.7.8

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	4.8061593686687588E-002	0.0000000000000000	2.8245278043260269E-002	0.0000000000000000
32	3.5100744469225829E-003	3.7753109531482321	2.0328473344334297E-003	3.7964359236009204
64	3.3735545105803333E-004	3.3791602511962950	1.9081122728303868E-004	3.4132839069983465
128	3.8748878888142008E-005	3.1220429984255471	2.1860049072119914E-005	3.1257775169093787
256	4.7445289664872518E-006	3.0298178001318887	2.6768668269783191E-006	3.0296793672274989
512	5.9027324975165167E-007	3.0068099968762390	3.3302802937562603E-007	3.0068298550228025
1024	7.3717697314358198E-008	3.0013000506268659	4.1590884736208977E-008	3.0013043271953017
2048	9.2150097932313977E-009	2.9999534026837824	5.1990112215349882E-009	2.9999582005915513
4096	1.1539634870520956E-009	2.9973881231725947	6.5099614587893484E-010	2.9975163618545317

表 53: $\frac{\Delta t}{\Delta x} = 0.6$, $T = 20$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	4.2967044151212865E-002	0.0000000000000000	2.5596690765722463E-002	0.0000000000000000
32	1.5515950437017254E-003	4.7914065549718199	9.2754014712292943E-004	4.7864037672566457
64	5.0360672810441602E-005	4.9453107049697005	2.8853942379170050E-005	5.0065693717551936
128	1.5915743517808822E-006	4.9837710969429496	9.0227834881329017E-007	4.9990520737099438
256	4.9834193294149403E-008	4.9971747704909300	2.8174638955213993E-008	5.0011035389095468
512	1.5531261874606382E-009	5.0038890197690336	8.7944462823230651E-010	5.0016605675258150
1024	5.7891375002200797E-011	4.7456828164537335	3.2684521755754758E-011	4.7499132525670467
2048	3.4750646272095997E-011	0.73630861454544394	1.8896439968330014E-011	0.79049312692510587

表 54: $\Delta t = \Delta x^{\frac{5}{3}}$, $T = 20$

9.2 Example 2—1D Burgers Equation

考虑凸 Hamiltonian[22]：

$$\begin{cases} \varphi_t + \frac{1}{2}\left(\varphi_x + 1\right)^2 = 0, \quad x \in [-1, 1] \\ \varphi(x, 0) = -\cos(\pi x) \\ \varphi(-1, t) = \varphi(1, t) \end{cases} \tag{134}$$

我们先来推导精确解。令 $v = \varphi_x + 1$，那么 v 满足 Burgers 方程：

$$\begin{cases} v_t + \frac{\partial}{\partial x}\frac{v^2}{2} = 0, \qquad\qquad x \in [-1, 1] \\ v(x, 0) = 1 + \pi \sin(\pi x) \\ v(-1, t) = v(1, t) \end{cases} \tag{135}$$

若记 $[0, 2\pi]$ 上以 $\sin x$ 为初值的标准 Burgers 方程的解为 $\Psi(x, t)$(前文已给出 Newton 迭代法求解的算法)。利用待定系数法可以解得，$[0, 2L]$ 上以 $a + b\sin(\frac{\pi}{L}x)$ 为初值的 Burgers 方程的解为：

$$\phi(x, t) = a + b\Psi\left(\frac{\pi}{L}\left(x - at\right), \frac{b\pi}{L}t\right) \tag{136}$$

[22]2021.7.8

在本问题中，代入 $L=1, a=1, b=\pi$ 得：

$$\varphi_x(x,t) = \pi\Psi\left(\pi(x-t), \pi^2 t\right) \tag{137}$$

我们在 (x,t) 处计算一次 $\varphi_x = v-1$ 的值，记 $x_0 = x - vt$，由 v 的特征线和原方程知：

$$\varphi_x(x_0+\tau v, \tau) = \varphi_x \tag{138}$$
$$\varphi_t(x_0+\tau v, \tau) = -\frac{1}{2}\left(\varphi_x+1\right)^2 \tag{139}$$

于是：

$$\begin{aligned}\varphi(x,t) =&\varphi(x_0,0) + \int_0^t (v,1)\cdot\left(\varphi_x,\varphi_t\right)(x_0+\tau v,\tau)\mathrm{d}\tau \\ =&-\cos(\pi x_0) + t\left(v(v-1) - \frac{1}{2}v^2\right) \\ =&-\cos(\pi(x-vt)) + tv\left(\frac{1}{2}v-1\right)\end{aligned} \tag{140}$$

同时，易见 φ_x 的间断在 $t=\frac{1}{\pi^2}$ 时刻出现。

数值计算时，因为 $H(u)=\frac{1}{2}(u+1)^2$，所以取 Lax-Friedrichs 分裂：

$$\widehat{H}\left(u^-,u^+\right) = \frac{1}{2}\left(\frac{u^-+u^+}{2}+1\right)^2 - \frac{1}{2}\max\left\{\left|u^-+1\right|, \left|u^++1\right|\right\}\left(u^+-u^-\right) \tag{141}$$

易见，$\varphi_x\in[-\pi,\pi]$，$\max\limits_{[-\pi,\pi]}|H'(u)| = \pi+1$，所以取 $\frac{\Delta t}{\Delta x}=\frac{0.6}{1+\pi}$，那么 CFL $c=0.6$.

首先，取 $T=\frac{0.5}{\pi^2}$，此时 φ_x 的间断尚未出现。

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	7.4082457243365111E-004	0.0000000000000000	1.3948438374029593E-003	0.0000000000000000
32	5.2328798682562507E-005	3.8234549078373345	1.0229914343229396E-004	3.7692376408975168
64	4.6702487108214175E-006	3.4860338524326782	9.4379328251514849E-006	3.4381793519094597
128	6.1195606845803509E-007	2.9319993891742002	1.3107522873756494E-006	2.8480758395904808
256	7.5448113133203079E-008	3.0198713611635251	1.6481679432356344E-007	2.9914599021796491
512	9.7080950549131696E-009	2.9582246815938462	2.1283992346354808E-008	2.9530225612328773
1024	1.2320529579649236E-009	2.9781239649570712	2.7029491977970110E-009	2.9771624882046290
2048	1.5520170596555861E-010	2.9888479487463795	3.4050673392016506E-010	2.9887791619093260
4096	1.9482178182618616E-011	2.9939175251911196	4.2742698269648827E-011	2.9939334387788148
8192	2.4487530072442662E-012	2.9920358159650231	5.3754778406300829E-012	2.9912130750717871
16384	3.5220586841151375E-013	2.7975564117348743	7.4051875742497941E-013	2.8597848207103800
32768	2.5018004260459752E-013	0.49345224014489064	3.2851499298658382E-013	1.1725770648780258

表 55: $\frac{\Delta t}{\Delta x}=\frac{0.6}{1+\pi}, T=\frac{0.5}{\pi^2}$

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	1.2380258048616359E-003	0.0000000000000000	2.4826336486772904E-003	0.0000000000000000
32	5.2328798682562507E-005	4.5642924362976496	1.0229914343229396E-004	4.6010054147409436
64	2.2072774983221130E-006	4.5672652178817419	4.1202907185011739E-006	4.6339041206471219
128	8.2797689321884850E-008	4.7365336997728944	1.7657022485795260E-007	4.5444321482501602
256	2.9487821950310753E-009	4.8113993361277432	8.9981431472541118E-009	4.2944709524876368
512	9.3285481491980511E-011	4.9823228904176542	2.8727853429444394E-010	4.9691052156967679
1024	2.5916565563988341E-012	5.1697061127910242	5.4072024635587468E-012	5.7314240105169132
2048	2.5221118048897846E-013	3.3611704097789432	3.5849101465146305E-013	3.9148735124467695
4096	6.9232489903977521E-013	-1.4568170031606198	9.6078700551061047E-013	-1.4222796798079105

表 56: $\Delta t = \Delta x^{\frac{5}{3}}$, $T = \frac{0.5}{\pi^2}$

我们再选取 $T = \frac{3.5}{\pi^2}$，此时有间断，我们绘制解的图像如下，其中标准 Burgers 解的迭代精度取为 10^{-16}. 可见，解没有出现振荡现象。

N	L^2 error	L^2 order	L^∞ error	L^∞ order
16	4.0200835307197032E-003	0.0000000000000000	9.7442936939031999E-003	0.0000000000000000
32	3.7005450805889834E-003	0.11948768695308019	1.4772886081637168E-002	-0.60032218151108430
64	7.2198708365587292E-004	2.3576928589399335	4.0735145739364681E-003	1.8586057285975941
128	4.4353652745874759E-004	0.70292010494337531	3.5477366996421178E-003	0.19937512783924100
256	1.4325269056545930E-004	1.6304906867265565	1.6197538899183628E-003	1.1311243194215983
512	3.9460339973251616E-005	1.8600869450034687	6.3128918180933857E-004	1.3594016890676390
1024	3.1862097691803981E-005	0.30856213364331131	7.2093101631452716E-004	-0.19156019065895546
2048	1.5605200303019990E-006	4.3517424716799784	3.5384052261819288E-005	4.3486900356847427
4096	8.1114171112866638E-007	0.94400098660908538	3.5668369741462258E-005	-1.1545999296847865E-002
8192	1.5419984772807879E-006	-0.92677545188873578	9.8682816040140953E-005	-1.4681536078794539
16384	7.3090782666830585E-008	4.3989680481235247	5.7783951892820262E-006	4.0940581058599950
32768	2.5049873068999382E-008	1.5448861891339718	2.5481947934918869E-006	1.1811933076107157

表 57: $\frac{\Delta t}{\Delta x} = \frac{0.6}{1+\pi}$, $T = \frac{3.5}{\pi^2}$

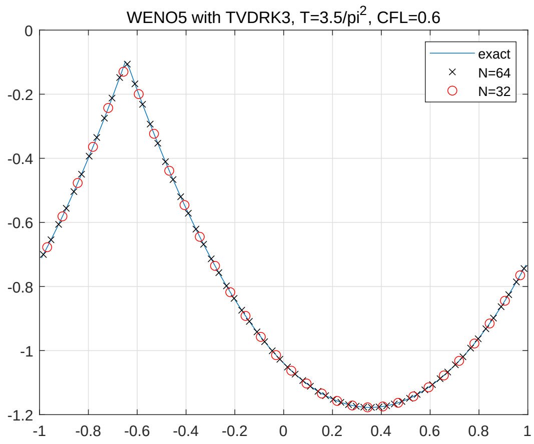

图 32: solution at $T = \frac{3.5}{\pi^2}$, $c = 0.6$

9.3 Example 3—1D Riemann Problem with a non-convex Hamiltonian

考虑非凸 Hamiltonian，Dirichlet 边界条件[23]：

$$\begin{cases} \varphi_t + \frac{1}{4}\left(\varphi_x^2 - 1\right)\left(\varphi_x^2 - 4\right) = 0, \quad x \in [-1, 1] \\ \varphi(x, 0) = -2\,|x| \\ \varphi(-1, t) = \varphi(1, t) = -2 \end{cases} \tag{142}$$

由于 $H(u) = \frac{1}{4}\left(u^2 - 1\right)\left(u^2 - 4\right)$，$H'(u) = u^3 - \frac{5}{2}u$，那么选用 Lax-Friedrichs 分裂：

$$\widehat{H}\left(u^-, u^+\right) = H\left(\frac{u^- + u^+}{2}\right) - \frac{1}{2}\max_{I(u^-,u^+)}\left|u^3 - \frac{5}{2}u\right|\left(u^+ - u^-\right) \tag{143}$$

注意到：$u^3 - \frac{5}{2}u$ 为奇函数，令 $\lambda = \sqrt{\frac{5}{6}}$，那么准确计算 $\alpha = \max\limits_{I(u^-,u^+)} |u^3 - \frac{5}{2}u|$ 如下：

1. 设 $a = \min\{u^-, u^+\}, b = \max\{u^-, u^+\}$.
2. a, b 同号，令 $a \leftarrow |a|, b \leftarrow |b|$，
   (a) a, b 在 λ 同侧：$\alpha = \max\left\{\left|a^3 - \frac{5}{2}a\right|, \left|b^3 - \frac{5}{2}b\right|\right\}$.
   (b) a, b 在 λ 异侧：$\alpha = \max\left\{\left|a^3 - \frac{5}{2}a\right|, \left|b^3 - \frac{5}{2}b\right|, \frac{5}{3}\lambda\right\}$.
3. a, b 异号：
   (a) $\max\{|a|, |b|\} \leqslant \lambda$：$\alpha = \max\left\{\left|a^3 - \frac{5}{2}a\right|, \left|b^3 - \frac{5}{2}b\right|\right\}$.
   (b) $\max\{|a|, |b|\} \geqslant \lambda$：$\alpha = \max\left\{\left|a^3 - \frac{5}{2}a\right|, \left|b^3 - \frac{5}{2}b\right|, \frac{5}{3}\lambda\right\}$.

下面，我们引入一个简易的处理 Dirichlet 边界的方法：

以左侧为例，由于使用 5 阶 WENO，我们设置 x_0 处的值为边界条件给出的值，然后在 $x_{0,1,2,3,4}$ 上对 u 构造 5 阶插值多项式，获得 u_{-1}, u_{-2}，其具有 5 阶误差。具体地，由 Lagrange 多项式插值知：

$$u_{-2} = 15u_0 - 40u_1 + 45u_2 - 24u_3 + 5u_4 \tag{144}$$

$$u_{-1} = 5u_0 - 10u_1 + 10u_2 - 5u_3 + u_4 \tag{145}$$

最后，因为 $|H'(u)| \leqslant 3$，所以取 $\frac{\Delta t}{\Delta x} = 0.2$，那么 CFL $c = 0.6$. 选取终止时刻 $T = 1.0$.

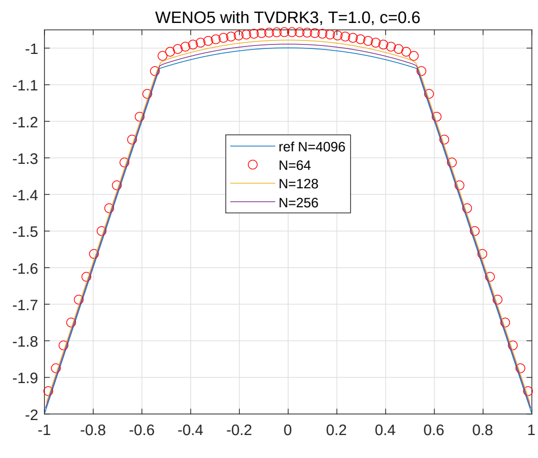

图 33: solution at $T = 1.0$, $c = 0.6$

[23] 2021.7.8

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