A High-Order Local Discontinuous Galerkin Method for the p-Laplace Equation

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Hightlights

1. Efficient preconditioner with hk-independent performance.
2. High-order error estimates assuming high regularity.
3. Estimates under a non-equivalent distance from existing ones.

Notations

- $\mathcal{T}_h$: a quasi-uniform simplicial mesh discretization of Ω.
- Γ^o, Γ^D, Γ^N: the unions of interior, Dirichlet and Neumann faces.
- $[\![\cdot]\!]$, $\{\{\cdot\}\}$: the DG jump and average operators on faces.
- Π_V: the L^2 projection onto a linear space V.
- Q_h, Σ_h, V_h: DG spaces for $\boldsymbol{q}$, $\boldsymbol{\sigma}$ and u (k-th order piecewise polynomial).
- $(\cdot;\cdot;\cdot)$, $\langle\cdot;\cdot;\cdot\rangle$: weighted L^2 inner product on d- and $d-1$-dimensional manifolds.

The p-Laplace equation

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $p \in (1,+\infty)$, and define $\mathcal{A}(\boldsymbol{\tau}) := |\boldsymbol{\tau}|^{p-2}\boldsymbol{\tau}$. The mixed and minimization forms of the p-Laplace equations are (1) and (2) respectively.

$$\begin{cases} \boldsymbol{q} - \nabla u = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{\sigma} - \mathcal{A}(\boldsymbol{q}) = \boldsymbol{0} & \text{in } \Omega, \\ -\nabla\cdot\boldsymbol{\sigma} = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma^D, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{g}_N\cdot\boldsymbol{n} & \text{on } \Gamma^N. \end{cases} \quad (1)$$

$$u \in \arg\min_{v\in V} J(v),$$
where $J(u) := \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - (f,u)_\Omega - \langle \boldsymbol{g}_N\cdot\boldsymbol{n}, u\rangle_{\Gamma^N}$
and $V = \left\{ v \in W^{1,p}(\Omega) : v|_{\Gamma^D} = g_D \right\}$ (2)

Spatial discretization: the LDG method

Define the LDG weak gradient operator $D_{DG} : W^{1,1}(\mathcal{T}_h) \times L^1(\Gamma^D) \mapsto Q_h$ to be: $\forall \boldsymbol{\zeta}_h \in \Sigma_h$,

$$(D_{DG}(v;g), \boldsymbol{\zeta}_h)_\Omega = (\nabla_h v, \boldsymbol{\zeta}_h)_\Omega - \langle [\![v]\!], \{\{\boldsymbol{\zeta}_h\}\} - \boldsymbol{C}_{12}[\![\boldsymbol{\zeta}_h]\!]\rangle_{\Gamma^o} - \langle v-g, \boldsymbol{\zeta}_h\cdot\boldsymbol{n}\rangle_{\Gamma^D}.$$

The LDG discretization of (1) for $(\boldsymbol{q}_h, \boldsymbol{\sigma}_h, u_h)$ is

$$\text{find } u_h \in V_h \text{ s.t. } \forall v_h \in V_h,\ J_h'(u_h)(v_h) = 0,$$
$$\boldsymbol{q}_h = D_{DG}(u_h; g_D),$$
$$\boldsymbol{\sigma}_h = \Pi_{\Sigma_h}\mathcal{A}(\boldsymbol{q}_h).$$

The ***equivalent*** and ***unisolvent*** minimization form for u_h (also a LDG discretization of (2)) is

$$u_h \in \arg\min_{v_h\in V_h} J_h(v_h). \quad (3)$$

Here, $J_h'(u_h)(v_h)$ is the Gâteaux derivative of $J_h(u_h)$ on v_h direction.

$$J_h(v_h) = \frac{1}{p}\|D_{DG}(v_h;g_D)\|^p_{L^p(\Omega)} + \frac{1}{p}\|[\![v_h]\!]\|^p_{L^p(\Gamma^o,\eta h_e^{1-p})} + \frac{1}{p}\|v_h - g_D\|^p_{L^p(\Gamma^D,\eta h_e^{1-p})} - (f,v_h)_\Omega - \langle \boldsymbol{g}_N\cdot\boldsymbol{n}, v_h\rangle_{\Gamma^N}.$$

$$J_h'(u_h)(v_h) = (\mathcal{A}(D_{DG}(u_h;g_D)), D_{DG}(v_h;0))_\Omega + \left\langle \eta\mathcal{A}\left(h_e^{-1}[\![u_h]\!]\right), [\![v_h]\!]\right\rangle_{\Gamma^o} + \left\langle \eta\mathcal{A}\left(h_e^{-1}(u_h-g_D)\boldsymbol{n}\right), v_h\boldsymbol{n}\right\rangle_{\Gamma^D} - (f,v_h)_\Omega - \langle \boldsymbol{g}_N\cdot\boldsymbol{n}, v_h\rangle_{\Gamma^N}.$$

Nonlinear solver: hk-independent preconditioned gradient descent

For J_h, the (unnormalized) steepest descent direction w_h at u_h under $\|\cdot\|$ (to be determined) is characterized by

$$J_h'(u_h)(w_h) = -\sup_{v_h\in V_h}\frac{J_h'(u_h)(v_h)}{\|v_h\|}\|w_h\|.$$

We choose the following linearized norm that is generalized from [Huang et al. 2007].

$$\|w_h\|^2 := \|D_{DG}(w_h;0)\|^2_{L^2(\Omega,|D_{DG}(u_h;g_D)|^{p-2})} + \|[\![w_h]\!]\|^2_{L^2(\Gamma^o,\eta h_e^{-1}|h_e^{-1}[\![u_h]\!]|^{p-2})} + \|w_h\|^2_{L^2(\Gamma^D,\eta h_e^{-1}|h_e^{-1}(u_h-g_D)|^{p-2})}.$$

Then the scheme for w_h (regularization terms in weights ignored) is an elliptic LDG scheme:

find $w_h \in V_h$ such that $\forall v_h \in V_h$:

$$\left(D_{DG}(w_h;0); |D_{DG}(u_h;g_D)|^{p-2}; D_{DG}(v_h;0)\right)_\Omega + \left\langle \eta h^{-1}[\![w_h]\!]; \left|h_e^{-1}[\![u_h]\!]\right|^{p-2}; [\![v_h]\!]\right\rangle_{\Gamma^o} + \left\langle \eta h_e^{-1}w_h; \left|h^{-1}(u_h-g_D)\right|^{p-2}; v_h\right\rangle_{\Gamma^D} = -J_h'(u_h)(v_h).$$

Theorem 1: Error estimates for the primal variable [Wu and Xu 2023]

Define $\|v\|_{J,p} := \left(\|D_{DG}(v;0)\|^p_{L^p(\Omega)} + \|[\![v]\!]\|^p_{L^p(\Gamma^o\cup\Gamma^D,\eta h_e^{1-p})}\right)^{\frac{1}{p}}$. *Assume* $\eta = \Theta(1)$ *and* $\boldsymbol{C}_{12} = \mathcal{O}(1)$. *Let* $(\boldsymbol{q},\boldsymbol{\sigma},u) \in W^{s,p}(\Omega) \times (W^{r,p'}(\Omega) \cap L^{p'}(\mathrm{div},\Omega)) \times W^{s+1,p}(\Omega)$ *be the exact solution. Assume WLOG that* $s, r \in \mathbb{N}$ *satisfy* $s \leqslant k$ *and* $r \leqslant k+1$. *Let* $u_h^* := \Pi_{V_h}u$, *then*

For $p \in (1,2]$:
$$\begin{cases} \|u_h - u\|_{L^p(\Omega)} \lesssim \|u_h - u_h^*\|_{J,p} + h^{s+1}|u|_{W^{s+1,p}(\Omega)}, \\ \|\boldsymbol{q}_h - \boldsymbol{q}\|_{L^p(\Omega)} + \|[\![u_h-u]\!]\|_{L^p(\Gamma^o\cup\Gamma^D,h^{1-p})} \lesssim \|u_h - u_h^*\|_{J,p} + h^s|u|_{W^{s+1,p}(\Omega)}, \\ \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{L^{p'}(\Omega)} \lesssim \|\boldsymbol{q}_h - \boldsymbol{q}\|^{p-1}_{L^p(\Omega)}, \\ \|u_h - u_h^*\|_{J,p} \lesssim C^{2-p}_{g_D,f,\boldsymbol{g}_N\cdot\boldsymbol{n}}\left(h^{s(p-1)}|u|^{p-1}_{W^{s+1,p}(\Omega)} + h^r|\boldsymbol{\sigma}|_{W^{r,p'}(\Omega)}\right). \end{cases}$$

For $p \in [2,+\infty)$:
$$\begin{cases} \|u_h - u\|_{L^p(\Omega)} \lesssim \|u_h - u_h^*\|_{J,p} + h^{s+1}|u|_{W^{s+1,p}(\Omega)}, \\ \|\boldsymbol{q}_h - \boldsymbol{q}\|_{L^p(\Omega)} + \|[\![u_h-u]\!]\|_{L^p(\Gamma^o\cup\Gamma^D,h^{1-p})} \lesssim \|u_h - u_h^*\|_{J,p} + h^s|u|_{W^{s+1,p}(\Omega)}, \\ \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{L^{p'}(\Omega)} \lesssim C^{p-2}_{g_D,f,\boldsymbol{g}_N\cdot\boldsymbol{n}}\|\boldsymbol{q}_h - \boldsymbol{q}\|_{L^p(\Omega)}, \\ \|u_h - u_h^*\|_{J,p} \lesssim h^{\frac{s}{p-1}}C^{\frac{p-2}{p-1}}_{g_D,f,\boldsymbol{g}_N\cdot\boldsymbol{n}}|u|^{\frac{1}{p-1}}_{W^{s+1,p}(\Omega)} + h^{\frac{r}{p-1}}|\boldsymbol{\sigma}|^{\frac{1}{p-1}}_{W^{r,p'}(\Omega)} + h^s|u|_{W^{s+1,p}(\Omega)}. \end{cases}$$

Here, the hidden constant $C > 0$ *is independent of* h. $C_{g_D,f,\boldsymbol{g}_N\cdot\boldsymbol{n}}$ *is defined to be*

$$C_{g_D,f,\boldsymbol{g}_N\cdot\boldsymbol{n}} := \|g_D\|_{W^{\frac{1}{p'},p}(\Gamma^D)} + \|f\|^{\frac{1}{p-1}}_{L^{p'}(\Omega)} + \|\boldsymbol{g}_N\cdot\boldsymbol{n}\|^{\frac{1}{p-1}}_{L^{p'}(\Gamma^N)}.$$

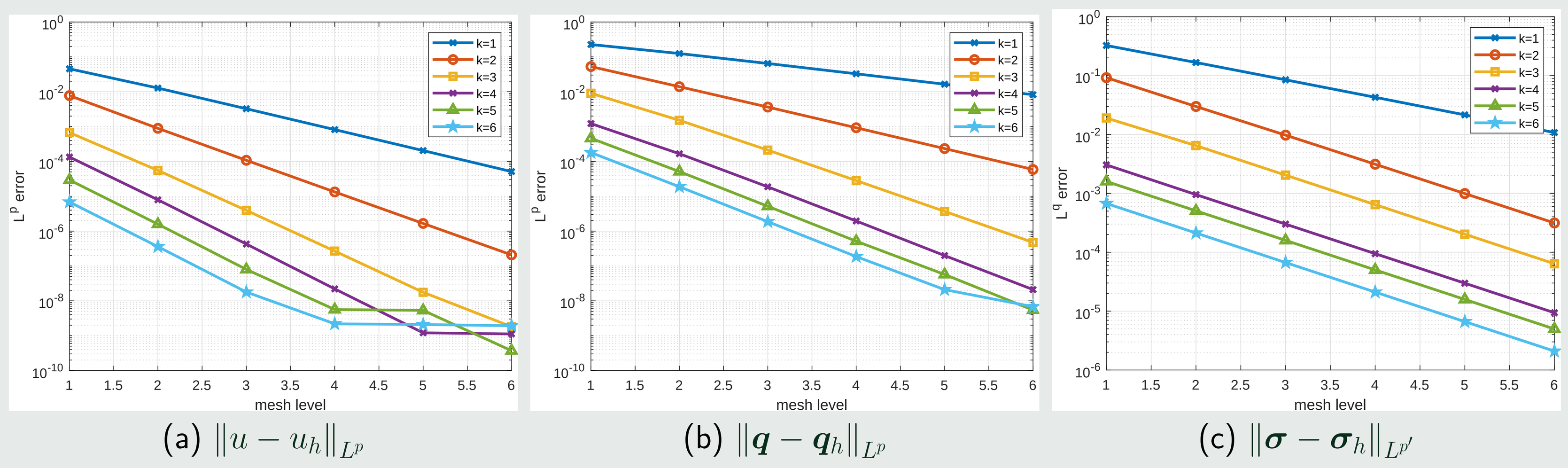
Numerical examples

Domain: $\Omega = \{(x,y) \in [-1,1]^2 : x - y \leqslant 1\}$. Let $r(x,y) = \sqrt{x^2+y^2}$. Take the exact solution [Barrett and Liu 1993]; [Cockburn and Shen 2016] as

$$u(x,y) = \frac{p-1}{(\sigma+2)^{\frac{1}{p-1}}}\,\frac{1 - r(x,y)^{\frac{\sigma+p}{p-1}}}{\sigma+p}.$$

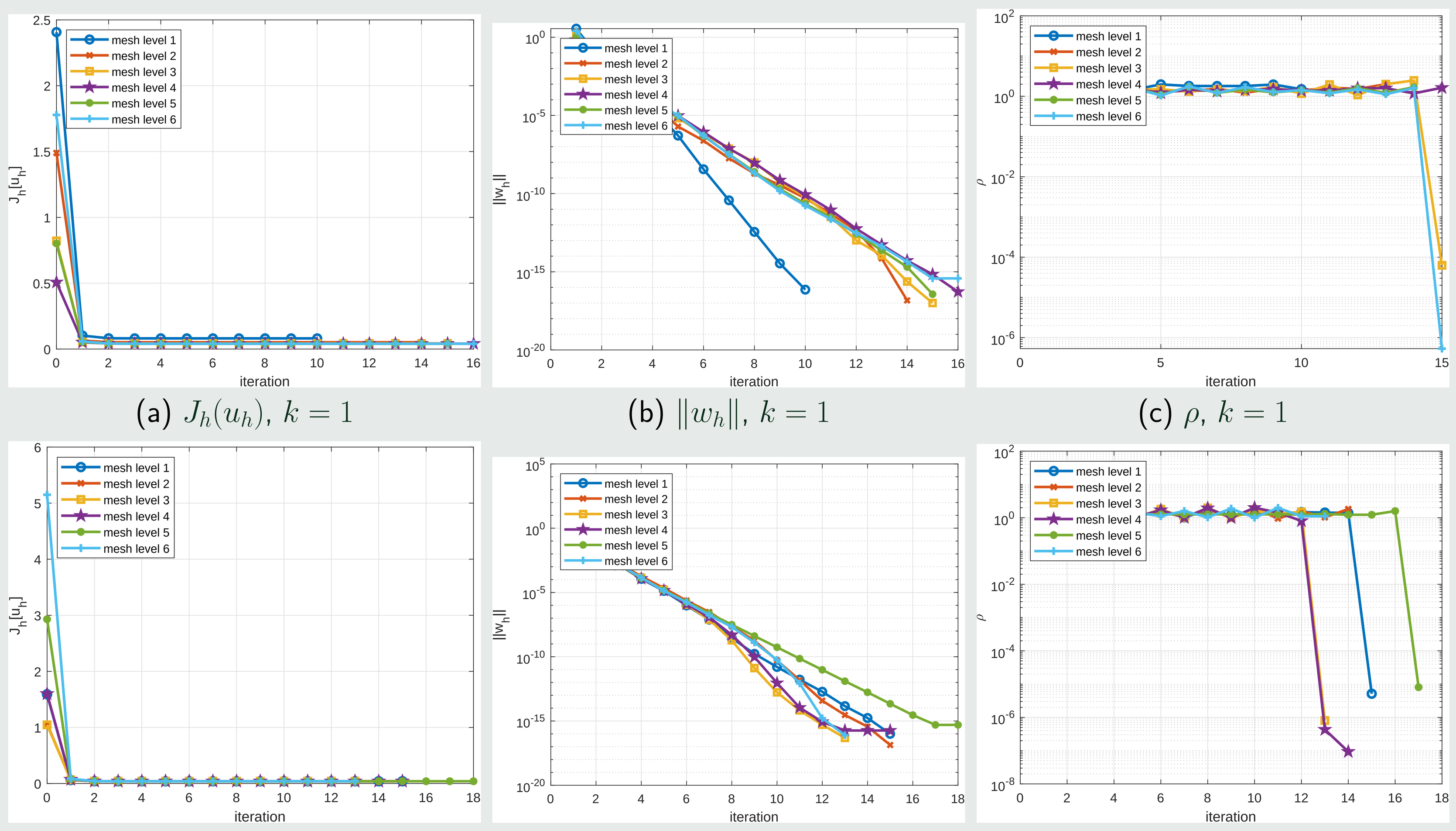
We consider two groups of parameters: $(\sigma,p) = (0,1.5)$ and $(\sigma,p) = (7,4)$.

Case 1: $(\sigma,p) = (0,1.5)$



(a) $\|u-u_h\|_{L^p}$ (b) $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^p}$ (c) $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^{p'}}$

Figure 1. Error convergence history. Parameters: $\sigma = 7$, $p = 4$.

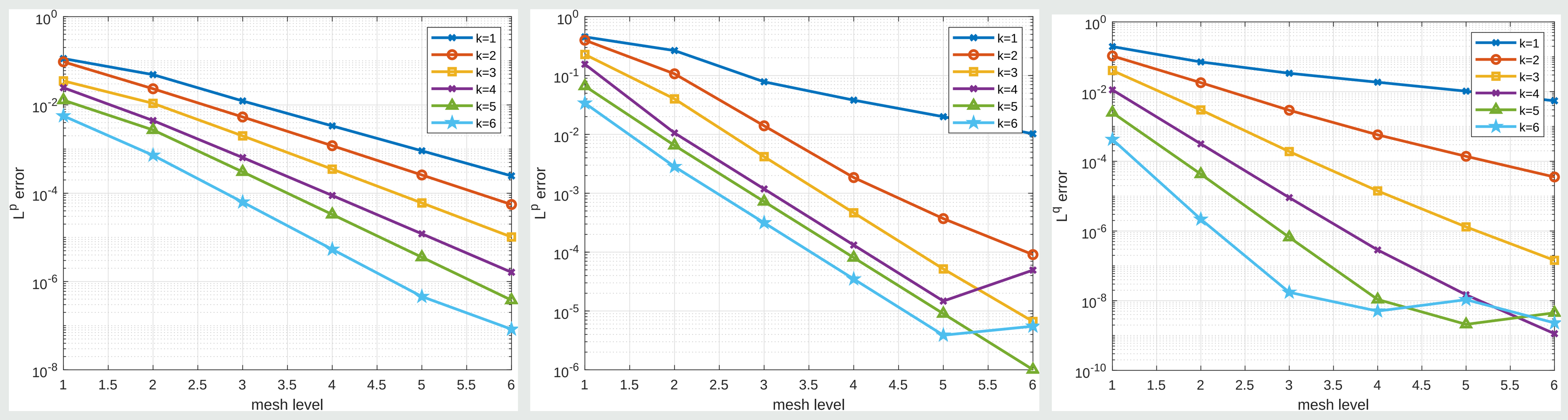


(a) $J_h(u_h)$, $k=1$ (b) $\|w_h\|$, $k=1$ (c) ρ, $k=1$

(d) $J_h(u_h)$, $k=6$ (e) $\|w_h\|$, $k=6$ (f) ρ, $k=6$

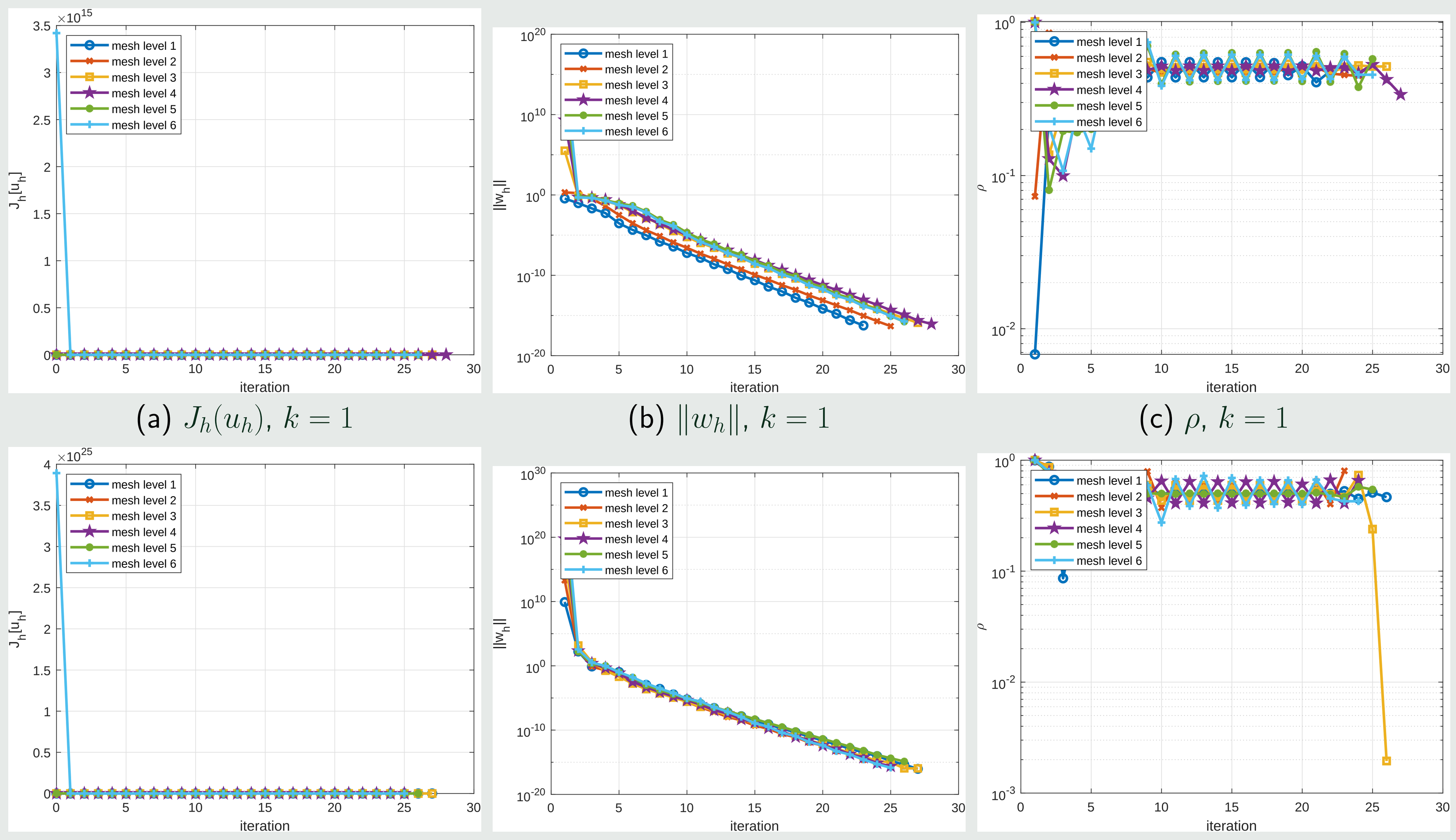
Figure 2. Gradient descent convergence history. Parameters: $\sigma = 0$, $p = 1.5$.

Case 2: $(\sigma,p) = (7,4)$



(a) $\|u-u_h\|_{L^p}$ (b) $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^p}$ (c) $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^{p'}}$

Figure 3. Error convergence history. Parameters: $\sigma = 7$, $p = 4$.



(a) $J_h(u_h)$, $k=1$ (b) $\|w_h\|$, $k=1$ (c) ρ, $k=1$

(d) $J_h(u_h)$, $k=6$ (e) $\|w_h\|$, $k=6$ (f) ρ, $k=6$

Figure 4. Gradient descent convergence history. Parameters: $\sigma = 7$, $p = 4$.

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